



# Derived Moduli of Complexes and Derived Grassmannians

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**Abstract** In the first part of this paper we construct a model structure for the category of filtered cochain complexes of modules over some associative ring  $R$  and explain how the classical Rees construction relates this to the usual projective model structure over cochain complexes. The second part of the paper is devoted to the study of derived moduli of sheaves: we give a new proof of the representability of the derived stack of perfect complexes over a proper scheme and then use the new model structure for filtered complexes to tackle moduli of filtered derived modules. As an application, we construct derived versions of Grassmannians and flag varieties.

**Keywords** Model category · Derived geometric stack · Lurie-Pridham representability · Filtered complex · Grassmannian · Rees functor

**Mathematics Subject Classifications (2010)** 14D20 · 14D23 · 18G55

## 1 Introduction

Recent developments in Derived Algebraic Geometry have lead many mathematicians to revise their approach to Moduli Theory: in particular one of the most striking results in this area is certainly *Lurie Representability Theorem* – proved by Lurie in 2004 as the main result of his PhD thesis [19] – which provides us with an explicit criterion to check

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whether a simplicial presheaf over some  $\infty$ -category of derived algebras gives rise to a derived geometric stack. Unfortunately the conditions in Lurie's result are often quite complicated to verify in concrete derived moduli problems involving algebro-geometrical structures, so for several years a rather narrow range of derived algebraic stacks have actually been constructed: in particular the most significant example known was probably the locally geometric derived stack of perfect complexes over a smooth proper scheme  $X$ , which was firstly studied by Toën and Vaquié in 2007 (see [40]) without using any representability result *à la* Lurie. Nonetheless a few years later Pridham developed in [26] several new representability criteria for derived geometric stacks which have revealed to be more suitable to tackle moduli problems arising in Algebraic Geometry, as he himself showed in [27] where he used such criteria to construct a variety of derived moduli stacks for schemes and (complexes of) sheaves. In [9] Halpern-Leistner and Preygel have also recovered Toën and Vaquié's result by using some generalisation of *Artin Representability Theorem* for ordinary algebraic stacks (see [1]), though their approach is not based on Pridham's theory, while in [23] Pandit generalised it to non-necessarily smooth schemes by studying the derived moduli stack of compact objects in a perfect symmetric monoidal infinity-category.

In this paper we give a third proof of existence and local geometricity of derived moduli for perfect complexes by means of Pridham's representability and then look at derived moduli of filtered perfect complexes: our main result is Theorem 3.33, which essentially shows that filtered perfect complexes of  $\mathcal{O}_X$ -modules – where  $X$  is a smooth and proper scheme – are parametrised by a locally geometric derived stack. In our strategy a key ingredient to tackle derived moduli of filtrations – in addition to Pridham's representability – is a good Homotopy Theory of filtered modules in complexes: for this reason the first part of this paper is devoted to construct a satisfying model structure on the category  $\mathfrak{FdgMod}_R$ , which is probably an interesting matter in itself. Theorem 2.18 shows that  $\mathfrak{FdgMod}_R$  is endowed with a natural cofibrantly generated model structure and Theorem 2.30 proves that this is nicely related to the standard projective model structure on  $\mathrm{dgMod}_R$  via the Rees construction. In the end, we conclude this paper by constructing derived versions of Grassmannians and flag varieties, which are obtained as suitable homotopy fibres of the derived stack of filtrations over the derived stack of complexes.

## 2 Homotopy Theory of Filtered Structures

This chapter is devoted to the construction of a good homotopy theory for filtered cochain complexes; for this reason we will first recall the standard projective model structure on cochain complexes and then use it to define a suitable one for filtered objects. At last, we will also study the Rees functor from a homotopy-theoretic viewpoint and see that it liaises coherently dg structures with filtered cochain ones.

### 2.1 Background on Model Categories

This section is devoted to review a few complementary definitions and results in Homotopy Theory which will be largely used in this paper; we will assume that the reader is familiar with the notions of model category, simplicial category and differential graded category: references for them include [6, 7, 10, 11, 29] and [38], while [8] provides a very clear and readable overview.

Let  $\mathfrak{C}$  be a complete and cocomplete category and  $I$  a class of morphisms in  $\mathfrak{C}$ ; recall from [11] that:

1. a map is  *$I$ -injective* if it has the right lifting property with respect to every map in  $I$  (denote by  $I$ -inj the class of  $I$ -injective morphisms in  $\mathfrak{C}$ );
2. a map is  *$I$ -projective* if it has the left lifting property with respect to every map in  $I$  (denote by  $I$ -proj the class of  $I$ -projective morphisms in  $\mathfrak{C}$ );
3. a map is an  *$I$ -cofibration* if it has the left lifting property with respect to every  $I$ -injective map (denote by  $I$ -cof the class of  $I$ -cofibrations in  $\mathfrak{C}$ );
4. a map is an  *$I$ -fibration* if it has the right lifting property with respect to every  $I$ -projective map (denote by  $I$ -fib the class of  $I$ -fibrations in  $\mathfrak{C}$ );
5. a map is a *relative  $I$ -cell complex* if it is a transfinite composition of pushouts of elements of  $I$  (denote by  $I$ -cell the class of  $I$ -cell complexes).

The above classes of morphisms satisfy a number of comparison relations: in particular we have that:

- $I$ -cof =  $(I$ -inj)-proj and  $I$ -fib =  $(I$ -proj)-inj;
- $I \subseteq I$ -cof and  $I \subseteq I$ -fib;
- $(I$ -cof)-inj =  $I$ -inj and  $(I$ -fib)-proj =  $I$ -proj;
- $I$ -cell  $\subseteq I$ -cof (see [11] Lemma 2.1.10);
- if  $I \subseteq J$  then  $I$ -inj  $\supseteq J$ -inj and  $I$ -proj  $\supseteq J$ -proj, thus  $I$ -cof  $\supseteq J$ -cof and  $I$ -fib  $\supseteq J$ -fib.

Fix some class  $S$  of morphisms in  $\mathfrak{C}$  and recall that an object  $A \in \mathfrak{C}$  is said to be *compact<sup>1</sup> relative to  $S$*  if for all sequences

$$C_0 \longrightarrow C_1 \longrightarrow \cdots \longrightarrow C_n \longrightarrow C_{n+1} \longrightarrow \cdots$$

such that each map  $C_n \rightarrow C_{n+1}$  is in  $S$ , the natural map

$$\lim_{\longrightarrow n} \operatorname{Hom}_{\mathfrak{C}}(A, C_n) \longrightarrow \operatorname{Hom}_{\mathfrak{C}}\left(A, \lim_{\longrightarrow n} C_n\right)$$

is an isomorphism; moreover  $A$  is said to be *compact* if it is compact relative to  $\mathfrak{C}$ .

**Definition 2.1** A model category  $\mathfrak{C}$  is said to be (*compactly*) *cofibrantly generated<sup>2</sup>* if there are sets  $I$  and  $J$  of maps such that:

1. the domains of the maps in  $I$  are compact relative to  $I$ -cell;
2. the domains of the maps in  $J$  are compact relative to  $J$ -cell;
3. the class of fibrations is  $J$ -inj;
4. the class of trivial fibrations is  $I$ -inj.

$I$  is said to be the set of *generating cofibrations*, while  $J$  is said to be the set of *generating trivial cofibrations*.

<sup>1</sup>In the language of [11] compact objects are called  $\aleph_0$ -small.

<sup>2</sup>The definition of cofibrantly generated model category as found in [11] Section 3.1 is slightly more general than the one provided by Definition 2.1, as it involves small objects rather than compact ones; anyway the proper definition requires some non-trivial Set Theory and moreover all examples we consider in this paper fit into the weaker notion determined by Definition 2.1, so we will stick to this.

Cofibrantly generated model categories are very useful as they come with a quite explicit characterisation of (trivial) fibrations and (trivial) cofibrations: this is exactly the content of the next result.

**Proposition 2.2** *Let  $\mathfrak{C}$  be a cofibrantly generated model category with  $I$  and  $J$  respectively being the set of generating cofibrations and generating trivial cofibrations. We have that:*

1. *the cofibrations form the class  $I\text{-cof}$ ;*
2. *every cofibration is a retract of a relative  $I$ -cell complex;*
3. *the domains of  $I$  are compact relative to the class of cofibrations;*
4. *the trivial cofibrations form the class  $J\text{-cof}$ ;*
5. *every trivial cofibration is a retract of a relative  $J$ -cell complex;*
6. *the domains of  $J$  are compact relative to the trivial cofibrations.*

*Proof* See [11] Proposition 2.1.18, which in turn relies on [11] Corollary 2.1.15 and [11] Proposition 2.1.16.  $\square$

The main reason we are interested in cofibrantly generated model categories is that they fit into a very powerful existence criterion – essentially due to Kan and Quillen and then developed by many more authors – which will be repeatedly used along this paper.

**Theorem 2.3** (Kan, Quillen) *Let  $\mathfrak{C}$  be a complete and cocomplete category and  $W$ ,  $I$ ,  $J$  three sets of maps. Then  $\mathfrak{C}$  is endowed with a cofibrantly generated model structure with  $W$  as the set of weak equivalences,  $I$  as the set of generating cofibrations and  $J$  as the set of generating trivial cofibrations if and only if:*

1. *the class  $W$  has the two-out-of-three property and is closed under retracts;*
2. *the domains of  $I$  are compact relative to  $I\text{-cell}$ ;*
3. *the domains of  $J$  are compact relative to  $J\text{-cell}$ ;*
4.  *$J\text{-cell} \subseteq W \cap I\text{-cof}$ ;*
5.  *$I\text{-inj} \subseteq W \cap J\text{-inj}$ ;*
6. *either  $W \cap I\text{-cof} \subseteq J\text{-cof}$  or  $W \cap J\text{-inj} \subseteq I\text{-inj}$ .*

*Proof* See [11] Theorem 2.1.19.  $\square$

Theorem 2.3 is a great tool in order to construct new model categories; moreover if we are given a cofibrantly generated model category we can often induce a good homotopy theory over other categories: this is the content of the following result, which again is essentially due to Kan and Quillen.

**Theorem 2.4** (Kan, Quillen) *Let  $\mathbf{F} : \mathfrak{C} \rightleftarrows \mathfrak{D} : \mathbf{G}$  be an adjoint pair of functors and assume  $\mathfrak{C}$  is a cofibrantly generated model category, with  $I$  as set of generating cofibrations,  $J$  as set of generating trivial fibrations and  $W$  as set of weak equivalences. Suppose further that:*

1.  *$\mathbf{G}$  preserves sequential colimits;*
2.  *$\mathbf{G}$  maps relative  $\mathbf{F}J\text{-cell}^3$  complexes to weak equivalences.*

<sup>3</sup>Of course, if  $S$  is a set of morphisms in  $\mathfrak{C}$ ,  $\mathbf{F}S$  will denote the set

$\mathbf{F}S := \{\mathbf{F}f \text{ s.t. } f \in S\}.$

Then the category  $\mathfrak{D}$  is endowed with a cofibrantly generated model structure where  $\mathbf{FI}$  is the set of generating cofibrations,  $\mathbf{FJ}$  is the set of generating trivial cofibrations and  $\mathbf{FW}$  as set of weak equivalences. Moreover  $(\mathbf{F}, \mathbf{G})$  is a Quillen pair with respect to these model structures.

*Proof* See [10] Theorem 11.3.2.  $\square$

The end of this section is devoted to review a famous comparison result due to Dold and Kan establishing an equivalence between the category of non-negatively graded chain complexes of  $k$ -vector spaces and that of simplicial  $k$ -vector spaces, which has very profound consequences in Homotopy Theory.

**Warning 2.5** Be aware that in the end of this section we will deal with (non-negatively graded) differential graded chain structures, while in the rest of the paper we will mostly be interested in cochain objects.

First of all, recall that the *normalisation* of a simplicial  $k$ -vector space  $(V, \partial_i, \sigma_j)$  is defined to be the non-negatively graded chain complex of  $k$ -vector spaces  $(\mathbf{NV}, \delta)$  where

$$(\mathbf{NV})_n := \bigcap_i \ker(\partial_i : V_n \rightarrow V_{n-1})$$

and  $\delta_n := (-1)^n \partial_n$ . Such a procedure defines a functor

$$\mathbf{N} : \mathfrak{sVect}_k \longrightarrow \mathfrak{Ch}_{\geq 0}(\mathfrak{Vect}_k).$$

On the other hand, let  $V$  be a chain complex of  $k$ -vector spaces and recall that its *denormalisation* is defined to be the simplicial vector space  $(\mathbf{KV}), \partial_i, \sigma_j)$  given in level  $n$  by the vector space

$$(\mathbf{KV})_n := \prod_{\substack{\eta \in \text{Hom}_{\Delta}([p], [n]) \\ \eta \text{ surjective}}} V_p[\eta] \quad (V_p[\eta] \simeq V_p).$$

**Remark 2.6** Notice that

$$(\mathbf{KV})_n \simeq V_0 \oplus V_1^{\oplus n} \oplus V_2^{\oplus \binom{n}{2}} \oplus \dots \oplus V_k^{\oplus \binom{n}{k}} \oplus \dots \oplus V_n^{\oplus \binom{n}{n}}.$$

In order to complete the definition of the denormalisation of  $V$  we need to define face and degeneracy maps: we will describe a combinatorial procedure to determine all of them. For all morphisms  $\alpha : [m] \rightarrow [n]$  in  $\Delta$ , we want to define a linear map  $\mathbf{K}(\alpha) : (\mathbf{KV})_n \rightarrow (\mathbf{KV})_m$ ; this will be done by describing all restrictions  $\mathbf{K}(\alpha, \eta) : V_p[\eta] \rightarrow (\mathbf{KV})_m$ , for any surjective non-decreasing map  $\eta \in \text{Hom}_{\Delta}([p], [n])$ .

For all such  $\eta$ , take the composite  $\eta \circ \alpha$  and consider its epi-monic factorisation<sup>4</sup>  $\epsilon \circ \eta'$ , as in the diagram

$$\begin{array}{ccc} [m] & \xrightarrow{\alpha} & [n] \\ \eta' \downarrow & & \downarrow \eta \\ [q] & \xrightarrow{\epsilon} & [p]. \end{array}$$

<sup>4</sup>The existence of such a decomposition is one of the key properties of the category  $\Delta$ .

Now

- if  $p = q$  (in which case  $\epsilon$  is just the identity map), then set  $\mathbf{K}(\alpha, \eta)$  to be the natural identification of  $V_p[\eta]$  with the summand  $V_p[\eta']$  in  $(\mathbf{K}V)_m$ ;
- if  $p = q + 1$  and  $\epsilon$  is the unique injective non-decreasing map from  $[p]$  to  $[p + 1]$  whose image misses  $p$ , then set  $\mathbf{K}(\alpha, \eta)$  to be the differential  $d_p : V_p \rightarrow V_{p-1}$ ;
- in all other cases set  $\mathbf{K}(\alpha, \eta)$  to be the zero map.

The above setting characterises all the structure of the simplicial vector space  $(\mathbf{K}V, \partial_i, \sigma_j)$ ; again, such a procedure defines a functor

$$\mathbf{K} : \mathcal{Ch}_{\geq 0}(\mathcal{Vect}_k) \longrightarrow \mathfrak{sVect}_k.$$

**Theorem 2.7** (Dold, Kan) *The functors  $\mathbf{N}$  and  $\mathbf{K}$  form an equivalence of categories between  $\mathfrak{sMod}_k$  and  $\mathcal{Ch}_{\geq 0}\mathfrak{Mod}_k$ .*

*Proof* See [7] Corollary 2.3 or [42] Theorem 8.4.1.  $\square$

The Dold-Kan correspondence described in Theorem 2.7 is known to induce a number of very interesting  $\infty$ -equivalences: for more details see for example [4, 7, 36, 42].

## 2.2 Homotopy Theory of Cochain Complexes

Fix an associative unital ring  $R$ : in this section we will review the standard model structure by which one usually endows the category of (unbounded) cochain complexes of  $R$ -modules, i.e. the so-called *projective model structure*; all the section is largely based on [11] Section 3.3, where the homotopy theory of chain complexes over an associative unital ring is extensively studied. Actually all results, constructions and proofs we are about to discuss herein are nothing but dual versions of the ones provided in [11]: we report them – adapted to cohomological conventions – for the reader’s convenience and because such arguments will be very important in the study of the Homotopy Theory of filtered complexes, which we will tackle in Section 2.3.

Recall that the category  $\mathfrak{dgMod}_R$  of *cochain complex of  $R$ -modules* (also referred as  $R$ -module in complexes) is one of the main examples of abelian category: as a matter of fact it is complete and cocomplete (limits and colimits are taken degreewise), the complex  $(0, 0)$  defined by the trivial module in each degree gives the zero object and short exact sequences are defined degreewise; for more details see [42].

Let  $(M, d) \in \mathfrak{dgMod}_R$ : as usual, we define its  $R$ -module of  $n$ -cocycles to be  $Z^n(M) := \ker(d^n)$ , its  $R$ -module of  $n$ -coboundaries to be  $B^n(M) := \operatorname{Im} d^{n-1} \leq Z^n(M)$  and its  $n^{\text{th}}$  cohomology  $R$ -module to be  $H^n(M) := Z^n(M)/B^n(M)$ ;  $(M, d)$  is said to be *acyclic* if  $H^n(M) = 0 \forall n \in \mathbb{Z}$ ; cocycles, coboundaries and cohomology define naturally functors from the category  $\mathfrak{dgMod}_R$  to the category  $\mathfrak{Mod}_R$  of  $R$ -modules. Finally, recall that a cochain map  $f : (M, d) \rightarrow (N, \delta)$  is said to be a *quasi-isomorphism* if it is a cohomology isomorphism, i.e. if  $H^n(f)$  is an isomorphism  $\forall n \in \mathbb{Z}$ . In the following, we will not explicitly mention the differential of a complex whenever it is clear from the context.

Now define the complexes

$$D_R(n) := \begin{cases} R & \text{if } k = n, n + 1 \\ 0 & \text{otherwise} \end{cases} \quad S_R(n) := \begin{cases} R & \text{if } k = n \\ 0 & \text{otherwise} \end{cases}$$

and the only non-trivial connecting map (the one between  $D_R(n)^n$  and  $D_R(n)^{n+1}$ ) is the identity.

**Remark 2.8** Observe that  $D_R(n)$  and  $S_R(n)$  are compact for all  $n$ .

**Theorem 2.9** Consider the sets

$$\begin{aligned} I_{\mathfrak{dgM}\mathfrak{o}\mathfrak{d}_R} &:= \{S_R(n+1) \rightarrow D_R(n)\}_{n \in \mathbb{Z}} \\ J_{\mathfrak{dgM}\mathfrak{o}\mathfrak{d}_R} &:= \{0 \rightarrow D_R(n)\}_{n \in \mathbb{Z}} \\ W_{\mathfrak{dgM}\mathfrak{o}\mathfrak{d}_R} &:= \{f : M \rightarrow N \mid f \text{ is a quasi-isomorphism}\}. \end{aligned} \quad (2.1)$$

The classes (2.1) define a cofibrantly generated model structure on  $\mathfrak{dgM}\mathfrak{o}\mathfrak{d}_R$ , where  $I_{\mathfrak{dgM}\mathfrak{o}\mathfrak{d}_R}$  is the set of generating cofibrations,  $J_{\mathfrak{dgM}\mathfrak{o}\mathfrak{d}_R}$  is the set of generating trivial cofibrations and  $W_{\mathfrak{dgM}\mathfrak{o}\mathfrak{d}_R}$  is the set of weak equivalences.

The proof of Theorem 2.9 (which corresponds to [11] Theorem 2.3.11) relies on Theorem 2.3, thus it amounts to explicitly describe fibrations, trivial fibrations, cofibrations and trivial cofibrations determined by the sets (2.1), which we do in the following propositions.

**Proposition 2.10**  $p \in \text{Hom}_{\mathfrak{dgM}\mathfrak{o}\mathfrak{d}_R}(M, N)$  is a fibration if and only if it is a degreewise surjection.

*Proof* This is dual to [11] Proposition 2.3.4. We want to characterise diagrams

$$\begin{array}{ccc} 0 & \longrightarrow & M \\ \downarrow & & \downarrow p \\ D_R(n) & \longrightarrow & N \end{array} \quad (2.2)$$

in  $\mathfrak{dgM}\mathfrak{o}\mathfrak{d}_R$  admitting a lifting. A diagram like (2.2) is equivalent to choosing an element  $y$  in  $N^n$ , while a lifting is equivalent to a pair  $(x, y) \in M^n \times N^n$  such that  $p^n(x) = y$ : it follows that  $p \in J_{\mathfrak{dgM}\mathfrak{o}\mathfrak{d}_R}\text{-inj}$  if and only if  $p^n$  is surjective for all  $n \in \mathbb{Z}$ .  $\square$

**Proposition 2.11**  $p \in \text{Hom}_{\mathfrak{dgM}\mathfrak{o}\mathfrak{d}_R}(M, N)$  is a trivial fibration if and only if it is in  $I_{\mathfrak{dgM}\mathfrak{o}\mathfrak{d}_R}\text{-inj}$ ; in particular  $W_{\mathfrak{dgM}\mathfrak{o}\mathfrak{d}_R} \cap J_{\mathfrak{dgM}\mathfrak{o}\mathfrak{d}_R}\text{-inj} = I_{\mathfrak{dgM}\mathfrak{o}\mathfrak{d}_R}\text{-inj}$ .

*Proof* This is dual to [11] Proposition 2.3.5. First of all, observe that any diagram in  $\mathfrak{dgM}\mathfrak{o}\mathfrak{d}_R$  of the form

$$\begin{array}{ccc} S_R(n+1) & \longrightarrow & M \\ \downarrow & & \downarrow p \\ D_R(n) & \longrightarrow & N \end{array} \quad (2.3)$$

is uniquely determined by an element in

$$X := \{(x, y) \in N^n \oplus Z^{n+1}(M) \mid p^{n+1}(y) = d^n(x)\}.$$

Moreover, there is a bijection between the set of diagrams like (2.3) admitting a lifting and

$$X' := \{(x, z, y) \in N^n \oplus M^n \oplus Z^{n+1}(M) \mid p^n(z) = x, d^n(z) = y, p^{n+1}(y) = d^n(x)\}.$$

Now suppose that  $p \in I\text{-inj}$ : we want to prove that it is degreewise surjective (because of Proposition 2.10) and a cohomology isomorphism. For any cocycle  $y \in Z^n(N)$ , the pair  $(y, 0)$  defines a diagram like (2.3), therefore, as  $p \in I\text{-inj}$ ,  $\exists z \in M^n$  such that  $p^n(z) = y$  and  $d^n(z) = 0$ , so the induced map  $Z^n(p) : Z^n(M) \rightarrow Z^n(N)$  is surjective; in particular the map  $H^n(p) : H^n(M) \rightarrow H^n(N)$  is surjective as well. We now show that  $p^n$  itself is

surjective: fix  $x \in N^n$  and consider  $d^n(x) \in Z^{n+1}(N)$ ; as the map  $Z^{n+1}(p)$  is surjective,  $\exists y \in Z^{n+1}(M)$  such that  $p^{n+1}(y) = d^n(x)$ , thus by the assumption  $\exists z \in M^n$  such that  $p^n(z) = x$ , hence  $p$  is a degreewise surjection. It remains to prove that  $H^n(p)$  is injective: fix  $x \in N^{n-1}$  and consider  $d^{n-1}(x) \in B^n(N) \leq Z^n(N)$ ; by the surjectivity of  $Z^n(p)$   $\exists y \in Z^n(M)$  such that  $p^n(y) = d^{n-1}(x)$ , so  $[y] \in \ker(H^n(p))$ . The pair  $(x, y)$  defines a diagram of the form (2.3), so the assumption on  $p$  implies the existence of  $z \in M^{n-1}$  such that  $d^{n-1}(z) = y$  and  $p^{n-1}(z) = x$ ; in particular, we have that  $\ker(H^n(p)) = 0$ , so  $H^n(p)$  is injective.

Now assume that  $p$  is a trivial fibration, i.e. a degreewise surjection with acyclic kernel; consider  $(x, y) \in X$ : we want to find  $z \in M^n$  such that  $(x, z, y) \in X'$ . The hypotheses on  $p$  are equivalent to the existence of a short exact sequence in  $\mathfrak{dgM}\mathfrak{o}\mathfrak{d}_R$

$$0 \longrightarrow K \longrightarrow M \xrightarrow{p} N \longrightarrow 0$$

such that  $H^n(K) = 0 \forall n \in \mathbb{Z}$ . Take any  $w \in M^n$  such that  $p^n(w) = x$ ; an immediate computation shows that  $dw - y \in Z^{n+1}(K)$  and, as  $K$  is acyclic,  $\exists v \in K^n$  such that  $dv = dw - y$ . Now define  $z := w - v$  and the result follows.  $\square$

The next step is describing cofibrations and trivial cofibrations generated by the sets (2.1), but we need to understand cofibrant objects in order to do that; in the following for any  $R$ -module  $P$  call  $D_R(n, P)$  the cochain complex defined by

$$D_R(n, P) := \begin{cases} P & \text{if } k = n, n+1 \\ 0 & \text{otherwise} \end{cases}$$

and in which the only non-trivial connecting map is the identity.

**Proposition 2.12** *If  $A \in \mathfrak{dgM}\mathfrak{o}\mathfrak{d}_R$  is cofibrant, then  $A^n$  is projective for all  $n$ . Conversely, any bounded above cochain complex of projective  $R$ -modules is cofibrant.*

*Proof* This is dual to [11] Lemma 2.3.6. Suppose  $A$  is a cofibrant object in  $\mathfrak{dgM}\mathfrak{o}\mathfrak{d}_R$  and let  $p : M \rightarrow N$  be a surjection between two  $R$ -modules; the  $R$ -linear map  $p : M \rightarrow N$  induces a morphism  $\dot{p} : D_R(n, M) \rightarrow D_R(n, N)$  (given by  $p$  itself in degree  $n$  and  $n-1$  and by the zero map elsewhere), which is immediately seen to be degreewise surjective with acyclic kernel, hence a trivial fibration by Proposition 2.11. Moreover any  $R$ -linear map  $f : A^n \rightarrow N$  defines a cochain morphism  $\dot{f} : A \rightarrow D_R(n, N)$  which is given by  $f$  in degree  $n+1$ ,  $fd$  in degree  $n$  and 0 elsewhere. By assumption the diagram in  $\mathfrak{dgM}\mathfrak{o}\mathfrak{d}_R$

$$\begin{array}{ccc} & D_R(n, M) & \\ \nearrow g & \downarrow \dot{p} & \\ A & \xrightarrow{\dot{f}} & D_R(n, N) \end{array}$$

admits a lifting  $g$ : now it suffices to look at the above diagram in degree  $n$  to see that  $A^n$  is projective.

Now suppose  $A$  is a bounded above cochain complex of projective  $R$ -modules (i.e.  $A^n = 0$  for  $n \gg 0$ ) and fix a trivial fibration in  $\mathfrak{dgM}\mathfrak{o}\mathfrak{d}_R$   $p : M \rightarrow N$  and a cochain map  $g : A \rightarrow N$ : we want to prove that  $g$  lifts to a morphism  $h \in \text{Hom}_{\mathfrak{dgM}\mathfrak{o}\mathfrak{d}_R}(A, M)$ , so we construct  $h^n$  by (reverse) induction. The base of the induction is guaranteed by the fact that  $A$  is assumed to be bounded above, so suppose that  $h^k$  has been defined for all  $k > n$ ; by Proposition 2.11  $p^n$  is surjective and has an acyclic kernel  $K$ , so since  $A^n$  is projective  $\exists f \in \text{Hom}_{\mathfrak{M}\mathfrak{o}\mathfrak{d}_R}(A^n, M^n)$  lifting  $g^n$ . Consider the  $R$ -linear map  $F : A^n \rightarrow M^{n+1}$  defined



as  $F := d^n f - h^{n+1} d^n$ , which measures how far  $f$  is to fit into a cochain map: an easy computation shows that  $p^{n+1} F = d^{n+1} F = 0$ , so  $F : A^n \rightarrow Z^{n+1}(K)$ , but, as  $K$  is acyclic, we get that  $F : A^n \rightarrow B^{n+1}(K)$ . Of course the map  $d^{n+1}$  gives a surjective  $R$ -linear morphism from  $K^n$  to  $B^{n+1}(K)$ , so by the projectiveness of  $A^n$   $F$  lifts to a map  $G \in \text{Hom}_{\text{dg}\mathcal{M}\text{od}_R}(A^n, K^n)$ . Now define  $h^n := f - G$  and the result follows.  $\square$

**Remark 2.13** A complex of projective  $R$ -modules is not necessarily cofibrant (get a counterexample by adapting [11] Remark 2.3.7); it is possible to give a complete characterisation of cofibrant objects in  $\text{dg}\mathcal{M}\text{od}_R$  in terms of *dg-projective complexes* (see [2]).

**Proposition 2.14**  $i \in \text{Hom}_{\text{dg}\mathcal{M}\text{od}_R}(M, N)$  is a cofibration if and only if it is a degreewise injection with cofibrant cokernel.

*Proof* This is dual to [11] Proposition 2.3.9. Suppose  $i$  is a cofibration, i.e. a map having the left lifting property with respect to degreewise surjections with acyclic kernel, by Proposition 2.11; there is an obvious morphism  $M \rightarrow D_R(n-1, M^n)$  given by  $d^{n-1}$  in degree  $n-1$  and the identity in degree  $n$ , while the canonical map  $D_R(n-1, M^n) \rightarrow 0$  is a trivial fibration, as  $D_R(n-1, M^n)$  is clearly acyclic. As a consequence we get a diagram in  $\text{dg}\mathcal{M}\text{od}_R$

$$\begin{array}{ccc} M & \longrightarrow & D_R(n-1, M^n) \\ i \downarrow & & \downarrow \\ N & \longrightarrow & 0 \end{array}$$

which admits a lifting as  $i$  is a cofibration; in particular this implies that  $i^n$  is an injection. At last recall that the class of cofibration in a model category is closed under pushouts: in particular  $0 \rightarrow \text{coker}(i)$  is a cofibration as it is the pushout of  $i$ , thus  $\text{coker}(i)$  is cofibrant.

Now suppose that  $i$  is a degreewise injection with cofibrant cokernel  $C$  and we are given a diagram of cochain complexes

$$\begin{array}{ccc} M & \xrightarrow{f} & X \\ i \downarrow & & \downarrow p \\ N & \xrightarrow{g} & Y \end{array} \quad (2.4)$$

where  $p$  is a degreewise surjection with acyclic kernel  $K$  (let  $j : K \rightarrow X$  be the kernel morphism): we want to construct a lifting in such a diagram. First of all notice that  $N^n \simeq M^n \oplus C^n$ , as  $C^n$  is projective by Proposition 2.12, so we have

$$\begin{aligned} d : N^n &\longrightarrow N^{n+1} \\ (x, z) &\longmapsto (d^n(x) + \tau^n(z), d^n(y)) \end{aligned}$$

where  $\tau^n : C^n \rightarrow A^{n+1}$  is some  $R$ -linear map such that  $d^n \tau^n = \tau^{n+1} d^n$ , and

$$\begin{aligned} g^n : N^n &\longrightarrow Y^n \\ (x, z) &\longmapsto p^n f^n(x) + \sigma^n(z) \end{aligned}$$

where  $\sigma^n : C^n \rightarrow Y^n$  satisfies the relation  $d^n \sigma^n = p^n f^n \tau^n + \sigma^{n+1} d^n$ . A lifting in the diagram (2.4) then consists of a collection  $\{v^n\}_{n \in \mathbb{Z}}$  of  $R$ -linear morphisms such that  $p^n v^n = \sigma^n$  and  $d^n v^n = v^{n+1} d^n + f^{n+1} \tau^n$ . As  $C^n$  is projective, fix  $G^n \in \text{Hom}_{\mathcal{M}\text{od}_R}(C^n, X^n)$  lifting  $\sigma^n$  and consider the map  $F^n : C^n \rightarrow X^{n+1}$  defined as  $F^n := d^n G^n - G^{n+1} d^n - f^{n+1} \tau^n$ . It is easily seen that  $p^{n+1} F^n = 0$  and  $d^{n+1} F^n = -d^{n+1} G^n + f^{n+1} \tau^n d^{n-1}$ , so there is an induced

cochain map  $s : C \rightarrow \Sigma K$ , where  $\Sigma K$  is the *suspension complex* defined by the relations  $(\Sigma K)^n = K^{n+1}$  and  $d_{\Sigma K} = -d_K$ . As  $K$  is acyclic, observe that  $s$  is cochain homotopic to 0 (see [11] Lemma 2.3.8 for details), thus there is  $h^n \in \text{Hom}_{\mathfrak{dg}\mathfrak{Mod}_R}(C^n, K^n)$  such that  $s = -d^n h^n + h^{n+1} d^n$ ; define  $v^n := G^n + j^n h^n$  and the result follows.  $\square$

**Proposition 2.15**  $i \in \text{Hom}_{\mathfrak{dg}\mathfrak{Mod}_R}(M, N)$  is in  $J_{\mathfrak{dg}\mathfrak{Mod}_R}\text{-cof}$  if and only if it is a degreewise injection with projective<sup>5</sup> cokernel; in particular  $J_{\mathfrak{dg}\mathfrak{Mod}_R}\text{-cof} \subseteq W_{\mathfrak{dg}\mathfrak{Mod}_R} \cap I_{\mathfrak{dg}\mathfrak{Mod}_R}\text{-cof}$ .

*Proof* This is dual to [11] Proposition 2.3.10. Suppose  $i \in J_{\mathfrak{dg}\mathfrak{Mod}_R}\text{-cof}$ , i.e. it has the left lifting property with respect to all fibrations; in particular it is a cofibration so by Proposition 2.14 it is a degreewise injection with cofibrant cokernel  $C$  and let  $c : N \rightarrow C$  be the cokernel morphism: we want to show that  $C$  is projective as a cochain complex. Fix a fibration  $p : X \rightarrow Y$  and consider the diagram

$$\begin{array}{ccc} M & \xrightarrow{0} & X \\ i \downarrow & & \downarrow p \\ N & \xrightarrow{fc} & Y \end{array} \quad (2.5)$$

where  $0 : M \rightarrow 0 \rightarrow X$  is the zero morphism and  $f \in \text{Hom}_{\mathfrak{dg}\mathfrak{Mod}_R}(C, N)$  is an arbitrary cochain map. By assumption diagram (2.5) admits a lifting, which is a cochain map  $h$  such that  $hi = 0$  and  $ph = fc$ ; it follows that  $h$  factors through a map  $g \in \text{Hom}_{\mathfrak{dg}\mathfrak{Mod}_R}(C, M)$  lifting  $f$ , so  $C$  is a projective cochain complex.

Now assume  $i$  is a degreewise injection with projective cokernel  $C$ : again, let  $c : N \rightarrow C$  denote the cokernel morphism and consider a diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & X \\ i \downarrow & & \downarrow p \\ N & \xrightarrow{g} & Y \end{array}$$

where  $p$  is a fibration, i.e. a degreewise surjection because of Proposition 2.10. Since  $C$  is projective, there is a retraction  $r : N \rightarrow M$  and it is easily seen that  $(pfr - g)i = 0$ , so the map  $pfr - g$  lifts to a map  $s \in \text{Hom}_{\mathfrak{dg}\mathfrak{Mod}_R}(C, Y)$ . Again, the projectiveness of  $C$  implies that there is a map  $t \in \text{Hom}_{\mathfrak{dg}\mathfrak{Mod}_R}(C, X)$  lifting  $s$ ; now the map  $fr - tc$  gives a lifting in diagram (2.5).

The last claim of the statement follows immediately by the fact that any projective cochain complex is also acyclic (see for example [11] or [42]).  $\square$

**Proposition 2.16** The set  $W_{\mathfrak{dg}\mathfrak{Mod}_R}$  of quasi-isomorphisms in  $\mathfrak{dg}\mathfrak{Mod}_R$  has the 2-out-of-3 property and is closed under retracts.

*Proof* This is a classical result in Homological Algebra: for a detailed proof see [12] Lemma 1.1 (apply it in cohomology).  $\square$

The above results (especially Proposition 2.11, Propositions 2.15 and 2.16) say that the category  $\mathfrak{dg}\mathfrak{Mod}_R$  endowed with the structure (2.1) fits into the hypotheses of Theorem 2.3, so Theorem 2.9 has been proved.

<sup>5</sup>Here projective means projective as a cochain complex.

Now assume  $R$  is a (possibly differential graded) commutative  $k$ -algebra, where  $k$  is a field of characteristic 0: under such hypothesis there is also a canonical simplicial enrichment on  $\mathfrak{dg}\mathfrak{M}\mathfrak{o}\mathfrak{d}_R$  (all the rest of the section is adapted from [27]).

For all  $R$ -modules in complexes  $M, N$  consider the chain complex  $(\mathrm{HOM}_{\mathfrak{dg}\mathfrak{M}\mathfrak{o}\mathfrak{d}_R}(M, N), \delta)$  defined by the relations

$$\begin{aligned} \mathrm{HOM}_{\mathfrak{dg}\mathfrak{M}\mathfrak{o}\mathfrak{d}_R}(M, N)_n &:= \mathrm{Hom}(M, N[-n]) \\ \forall f \in \mathrm{HOM}_{\mathfrak{dg}\mathfrak{M}\mathfrak{o}\mathfrak{d}_R}(M, N)_n \quad \delta_n(f) &:= \bar{d}^n \circ f - (-1)^n f \circ d^n \in \mathrm{HOM}_{\mathfrak{dg}\mathfrak{M}\mathfrak{o}\mathfrak{d}_R}(M, N)_{n-1}. \end{aligned} \quad (2.6)$$

Formulae (2.6) make  $\mathfrak{dg}\mathfrak{M}\mathfrak{o}\mathfrak{d}_R$  into a differential graded category over  $k$ , thus the simplicial structure on  $\mathfrak{dg}\mathfrak{M}\mathfrak{o}\mathfrak{d}_R$  will be given by setting

$$\underline{\mathrm{Hom}}_{\mathfrak{dg}\mathfrak{M}\mathfrak{o}\mathfrak{d}_R}(M, N) := \mathbf{K}(\tau_{\geq 0} \mathrm{HOM}_{\mathfrak{dg}\mathfrak{M}\mathfrak{o}\mathfrak{d}_R}(M, N)) \quad (2.7)$$

where  $\mathbf{K}$  is the simplicial denormalisation functor giving the Dold-Kan correspondence (see Section 2.1) and  $\tau_{\geq 0}$  is good truncation.

## 2.3 Homotopy Theory of Filtered Cochain Complexes

Let  $R$  be any associative unital ring: in this section we will endow the category of filtered cochain complexes with a model structure which turns to be compatible (in a sense which will be clarified in Section 1.6) with the projective model structure on  $\mathfrak{dg}\mathfrak{M}\mathfrak{o}\mathfrak{d}_R$ .

Recall that a *filtered cochain complex of  $R$ -modules* (also referred as *filtered  $R$ -module in complexes*) consists of a pair  $(M, F)$ , where  $M \in \mathfrak{dg}\mathfrak{M}\mathfrak{o}\mathfrak{d}_R$  and  $F$  is a decreasing filtration on it, i.e. a collection  $\{F^k M\}_{k \in \mathbb{N}}$  of subcomplexes of  $M$  such that  $F^{k+1} M \subseteq F^k M$  and  $F^0 M = M$ ; as a consequence an object  $(M, F) \in \mathfrak{F}\mathfrak{dg}\mathfrak{M}\mathfrak{o}\mathfrak{d}_R$  looks like a diagram of the form

$$\begin{array}{ccccccc} \cdots & \longrightarrow & M^{n-1} & \longrightarrow & M^n & \longrightarrow & M^{n+1} \longrightarrow \cdots \\ & & \uparrow & & \uparrow & & \uparrow \\ \cdots & \longrightarrow & F^1 M^{n-1} & \longrightarrow & F^1 M^n & \longrightarrow & F^1 M^{n+1} \longrightarrow \cdots \\ & & \uparrow & & \uparrow & & \uparrow \\ \cdots & \longrightarrow & F^2 M^{n-1} & \longrightarrow & F^2 M^n & \longrightarrow & F^2 M^{n+1} \longrightarrow \cdots \\ & & \uparrow & & \uparrow & & \uparrow \\ & & \vdots & & \vdots & & \vdots \end{array}$$

A morphism of filtered complexes is a cochain map preserving filtrations, so denote by  $\mathfrak{F}\mathfrak{dg}\mathfrak{M}\mathfrak{o}\mathfrak{d}_R$  the category made of filtered  $R$ -modules in complexes and their morphisms.

The category  $\mathfrak{F}\mathfrak{dg}\mathfrak{M}\mathfrak{o}\mathfrak{d}_R$  is both complete and cocomplete: as a matter of fact let  $(M_\alpha, F_\alpha)_{\alpha \in I}$  and  $(N_\beta, F_\beta)$  be respectively an inverse system and a direct system in  $\mathfrak{F}\mathfrak{dg}\mathfrak{M}\mathfrak{o}\mathfrak{d}_R$ : we have that

$$\begin{aligned} \varinjlim_\alpha (M_\alpha, F_\alpha) &= (M, F) \quad \text{where } F^k M := \varinjlim_\alpha F_\alpha^k M_\alpha \\ \varprojlim_\beta (N_\beta, F_\beta) &= (N, F) \quad \text{where } F^k N := \varprojlim_\beta F_\beta^k N_\beta. \end{aligned}$$

In particular the filtered complex  $(0, T)$ , where 0 is the zero cochain complex and  $T$  is the trivial filtration over it, is the zero object of the category  $\mathfrak{F}\mathfrak{dg}\mathfrak{M}\mathfrak{o}\mathfrak{d}_R$ .

Define the filtered complexes

$$\begin{aligned} (D_R(n, p), F) \quad \text{where} \quad F^k D_R(n, p) &:= \begin{cases} D_R(n) & \text{if } k \leq p \\ 0 & \text{otherwise} \end{cases} \\ (S_R(n, p), F) \quad \text{where} \quad F^k S_R(n, p) &:= \begin{cases} S_R(n) & \text{if } k \leq p \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

**Remark 2.17** Observe that  $(D_R(n, p), F)$  and  $(S_R(n, p), F)$  are compact for all  $n$  and all  $p$ .

In the following we will sometimes drop explicit references to filtrations if the context makes them clear.

**Theorem 2.18** *Consider the sets*

$$\begin{aligned} I_{\mathfrak{F}\mathfrak{d}\mathfrak{g}\mathfrak{M}\mathfrak{o}\mathfrak{d}_R} &:= \{S_R(n+1, p) \rightarrow D_R(n, p)\}_{n \in \mathbb{Z}} \\ J_{\mathfrak{F}\mathfrak{d}\mathfrak{g}\mathfrak{M}\mathfrak{o}\mathfrak{d}_R} &:= \{0 \rightarrow D_R(n, p)\}_{n \in \mathbb{Z}} \end{aligned}$$

$$W_{\mathfrak{F}\mathfrak{d}\mathfrak{g}\mathfrak{M}\mathfrak{o}\mathfrak{d}_R} := \{f: (M, F) \rightarrow (N, F) \mid H^n(F^p f) \text{ is an isomorphism } \forall n \in \mathbb{Z}, \forall p \in \mathbb{N}\}. \quad (2.8)$$

The classes (2.8) define a cofibrantly generated model structure on  $\mathfrak{F}\mathfrak{d}\mathfrak{g}\mathfrak{M}\mathfrak{o}\mathfrak{d}_R$ , where  $I_{\mathfrak{F}\mathfrak{d}\mathfrak{g}\mathfrak{M}\mathfrak{o}\mathfrak{d}_R}$  is the set of generating cofibrations,  $J_{\mathfrak{F}\mathfrak{d}\mathfrak{g}\mathfrak{M}\mathfrak{o}\mathfrak{d}_R}$  is the set of generating trivial cofibrations and  $W_{\mathfrak{F}\mathfrak{d}\mathfrak{g}\mathfrak{M}\mathfrak{o}\mathfrak{d}_R}$  is the set of weak equivalences.

As done in Section 2.2, proving Theorem 2.18 amounts to provide a precise description of fibrations, trivial fibrations, cofibrations and trivial cofibrations determined by the sets (2.8), which we do in the following propositions.

**Proposition 2.19**  $p \in \text{Hom}_{\mathfrak{F}\mathfrak{d}\mathfrak{g}\mathfrak{M}\mathfrak{o}\mathfrak{d}_R}((M, F), (N, F))$  is a fibration if and only if  $F^k p^n$  is surjective  $\forall k \in \mathbb{N}, \forall n \in \mathbb{Z}$ .

*Proof* We want to characterise diagrams

$$\begin{array}{ccc} 0 & \longrightarrow & (M, F) \\ \downarrow & & \downarrow p \\ D_R(n, p) & \longrightarrow & (N, F) \end{array} \quad (2.9)$$

in  $\mathfrak{F}\mathfrak{d}\mathfrak{g}\mathfrak{M}\mathfrak{o}\mathfrak{d}_R$  admitting a lifting. A diagram like (2.9) corresponds to the sequence of diagrams in  $\mathfrak{d}\mathfrak{g}\mathfrak{M}\mathfrak{o}\mathfrak{d}_R$

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & & 0 & \longrightarrow & F^1 M & \cdots & 0 & \longrightarrow & F^p M & & 0 & \longrightarrow & F^{p+1} M \\ \downarrow & & \downarrow p & & \downarrow & & \downarrow F^1 p & & \downarrow & & \downarrow F^p p & & \downarrow & & \downarrow F^{p+1} p \\ D_R(n) & \longrightarrow & N & & D_R(n) & \longrightarrow & F^1 N & & D_R(n) & \longrightarrow & F^p N & & 0 & \longrightarrow & F^{p+1} N \end{array} \quad (2.10)$$

and – as we did in the proof of Proposition 2.10 – we see that the sequence (2.10) corresponds bijectively to an element  $(x^0, x^1, \dots, x^p) \in N^n \times F^1 N^n \times \dots \times F^p N^n$ , where  $x^p \in F^p N^n$  determines  $x^k \in F^k N^n$  for all  $k \leq p$  through the inclusion maps defining the filtration  $F$ . Again, Proposition 2.10 ensures that a diagram like (2.9) admits a lifting if and only if maps  $F^k p^n$  are surjective  $\forall k \leq p$  if and only if the map  $F^p p^n$  is surjective

(as observed above, what happens in level  $p$  determines the picture in lower levels), so the result follows letting  $n$  and  $p$  vary.  $\square$

**Proposition 2.20**  $p \in \text{Hom}_{\mathfrak{D}\mathfrak{G}_R}((M, F), (N, F))$  is a trivial fibration if and only if  $F^k p$  is degreewise surjective with acyclic kernel; in particular  $W_{\mathfrak{F}\mathfrak{d}\mathfrak{g}\mathfrak{M}\mathfrak{o}\mathfrak{d}_R} \cap J_{\mathfrak{F}\mathfrak{d}\mathfrak{g}\mathfrak{M}\mathfrak{o}\mathfrak{d}_R} \text{-inj} = I_{\mathfrak{F}\mathfrak{d}\mathfrak{g}\mathfrak{M}\mathfrak{o}\mathfrak{d}_R} \text{-inj}$ .

*Proof* We want to characterise diagrams

$$\begin{array}{ccc} S_R(n+1, p) & \longrightarrow & (M, F) \\ \downarrow & & \downarrow p \\ D_R(n, p) & \longrightarrow & (N, F) \end{array} \quad (2.11)$$

in  $\mathfrak{F}\mathfrak{d}\mathfrak{g}\mathfrak{M}\mathfrak{o}\mathfrak{d}_R$  admitting a lifting. A diagram like (2.9) corresponds to the sequence of diagrams in  $\mathfrak{d}\mathfrak{g}\mathfrak{M}\mathfrak{o}\mathfrak{d}_R$

$$\begin{array}{ccccccc} S_R(n+1) & \longrightarrow & M & & S_R(n+1) & \longrightarrow & F^1 M & \cdots & S_R(n+1) & \longrightarrow & F^p M & & 0 & \longrightarrow & F^{p+1} M \\ \downarrow & & \downarrow p & & \downarrow & & \downarrow F^1 p & & \downarrow & & \downarrow F^p p & & \downarrow & & \downarrow F^{p+1} p \\ D_R(n) & \longrightarrow & N & & D_R(n) & \longrightarrow & F^1 N & & D_R(n) & \longrightarrow & F^p N & & 0 & \longrightarrow & F^{p+1} N \end{array} \quad (2.12)$$

and – as we did in the proof of Proposition 2.11 – we see that the sequence (2.12) corresponds bijectively to an element  $((x_0, y_0), (x_1, y_1), \dots, (x_p, y_p)) \in X_0 \times X_1 \times \dots \times X_p$ , where

$$X_k := \left\{ (x_k, y_k) \in F^k N^n \oplus F^k Z^{n+1}(M) \mid F^k p^{n+1}(y_k) = F^k d^n(x_k) \right\}$$

and moreover the pair  $(x_p, y_p)$  determines all the previous ones through the inclusion maps defining the filtration  $F$ . Now by Proposition 2.11 a diagram like (2.11) admits a lifting if and only if  $F^p p$  is degreewise surjective and induces an isomorphism in cohomology, thus the result follows letting  $n$  and  $p$  vary.  $\square$

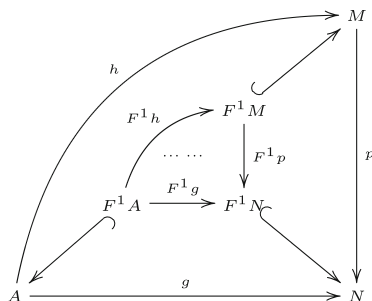
As done in Section 2.2, we study cofibrant objects defined by the structure (2.8).

**Proposition 2.21** Let  $(A, F)$  be a filtered complex of  $R$ -modules. If  $(A, F)$  is cofibrant then  $F^k A^n$  is a projective  $R$ -module  $\forall n \in \mathbb{Z}, \forall k \in \mathbb{N}$ ; conversely if  $F^k A$  is cofibrant as an object in  $\mathfrak{d}\mathfrak{g}\mathfrak{M}\mathfrak{o}\mathfrak{d}_R$  and the filtration  $F$  is bounded above then  $(A, F)$  is cofibrant.

*Proof* Suppose  $(A, F)$  is cofibrant and consider a trivial fibration  $p \in \text{Hom}_{\mathfrak{F}\mathfrak{d}\mathfrak{g}\mathfrak{M}\mathfrak{o}\mathfrak{d}_R}((M, F), (N, F))$  and any morphism  $g \in \text{Hom}_{\mathfrak{F}\mathfrak{d}\mathfrak{g}\mathfrak{M}\mathfrak{o}\mathfrak{d}_R}((A, F), (N, F))$ . By assumption, there exists a morphism  $h$  lifting  $g$ , so the diagram

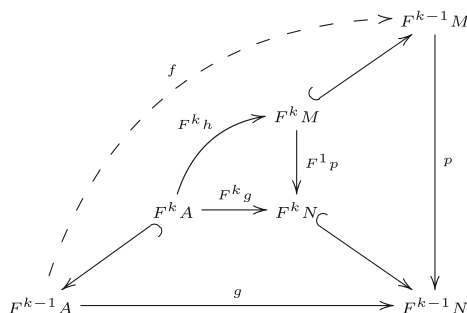
$$\begin{array}{ccc} & & (M, F) \\ & \nearrow h & \downarrow p \\ (A, F) & \xrightarrow{g} & (N, F) \end{array}$$

commutes. In particular this means that the big diagram



in  $\mathfrak{dgMod}_R$  commutes; now it suffices to apply Proposition 2.12 to show that  $F^k A^n$  is a projective  $R$ -module  $\forall n \in \mathbb{Z}, \forall k \in \mathbb{N}$ .

Now assume that  $F^k A$  is a cofibrant cochain complex (which in particular implies that  $F^k A^n$  is a projective  $R$ -module  $\forall n \in \mathbb{Z}$  by Proposition 2.12) and  $F$  is bounded above: we want to prove that  $(A, F)$  is cofibrant as a filtered module in complexes. Let  $p \in \text{Hom}_{\mathfrak{FdgMod}_R}((M, F), (N, F))$  be a trivial fibration and pick a morphism  $g \in \text{Hom}_{\mathfrak{FdgMod}_R}(A, M)$ : we want to show that there is a morphism  $h$  lifting  $g$ . By reverse induction, assume that  $F^p h : F^p A \rightarrow F^p M$  has been defined for all  $p \geq k$  (the boundedness of  $F$  ensures that we can get started): we want to construct a lifting in level  $k - 1$ . Consider the diagram



and observe that a lifting  $f \in \text{Hom}_{\mathfrak{dgMod}_R}(F^{k-1} A, F^{k-1} M)$  does exist because  $F^{k-1} A$  is cofibrant as an object in  $\mathfrak{dgMod}_R$ ; moreover since  $F^{k-1} A$  is projective in each degree we are allowed to choose  $f$  such that  $f|_{F^k A} = F^k h$ , thus the result follows.  $\square$

**Remark 2.22** The assumption on the filtration in Proposition 2.21 is probably too strong: it can be substituted with any hypothesis providing a base for the above inductive argument.

**Proposition 2.23** *There is an inclusion  $J_{\mathfrak{FdgMod}_R}\text{-cof} \subseteq W_{\mathfrak{FdgMod}_R} \cap I_{\mathfrak{FdgMod}_R}\text{-cof}$ .*

*Proof* Suppose  $i \in \text{Hom}_{\mathfrak{FdgMod}_R}((M, F), (N, F))$  is a  $J_{\mathfrak{FdgMod}_R}$ -cofibration, i.e. it has the left lifting property with respect to fibrations; in particular it lies in  $I_{\mathfrak{FdgMod}_R}\text{-cof}$ , so we only need to prove that  $H^n(F^k i)$  is an isomorphism  $\forall n \in \mathbb{Z}, \forall k \in \mathbb{N}$ . Let  $p \in$

$\mathrm{Hom}_{\mathfrak{FdgMod}_R}((X, F), (Y, F))$  be any fibration, so by Proposition 2.19  $F^k p^n$  is surjective  $\forall n \in \mathbb{Z}, \forall k \in \mathbb{N}$ : by assumption the diagram

$$\begin{array}{ccc} (M, F) & \longrightarrow & (X, F) \\ \downarrow i & & \downarrow p \\ (N, F) & \longrightarrow & (Y, F) \end{array}$$

admits a lifting and, unfolding it, we get that the diagram in  $\mathrm{dgMod}_R$

$$\begin{array}{ccc} F^k M & \longrightarrow & X \\ \downarrow F^k i & & \downarrow F^k p \\ F^k N & \longrightarrow & F^k Y \end{array}$$

lifts as well. Letting  $p$  vary among all fibrations in  $\mathfrak{FdgMod}_R$  we see that  $F^k i$  has the right lifting property with respect to all degreewise surjections in  $\mathrm{dgMod}_R$ , so by Proposition 2.10 and Proposition 2.15 it is a trivial cofibration in  $\mathrm{dgMod}_R$ ; in particular this means that  $H^n(F^k i)$  is an isomorphism  $\forall n \in \mathbb{Z}, \forall k \in \mathbb{N}$ , so the result follows.  $\square$

**Proposition 2.24** *The set  $W_{\mathfrak{FdgMod}_R}$  has the 2-out-of-3 property and is closed under retracts.*

*Proof* The result follows immediately by applying Proposition 2.16 levelwise in the filtration.  $\square$

The above results (especially Propositions 2.20, 2.23 and 2.24) say that the category  $\mathfrak{FdgMod}_R$  endowed with the structure (2.8) fits into the hypotheses of Theorem 2.3, so Theorem 2.18 has been proved.

**Remark 2.25** We have not provided a complete description of cofibrations as this is not really needed in order to establish that data (2.8) endow  $\mathfrak{FdgMod}_R$  with a model structure; clearly all morphism  $f : (M, F) \rightarrow (N, F)$  for which  $F^k f : F^k M \rightarrow F^k N$  is a cofibration in  $\mathrm{dgMod}_R$  for all  $k$  are cofibrations for such model structure, but it is not clear (nor expected) that these are all of them. Actually we believe that a careful characterisation of cofibrations should be quite complicated.

**Remark 2.26** In the definition of the category  $\mathfrak{FdgMod}_R$  we have assumed all filtrations to be indexed by natural numbers, i.e. to be bounded below. A natural question would be whether the model structure determined by Theorem 2.18 extends to filtered complexes whose filtrations are allowed to be infinite in both directions. The answer is positive: Theorem 2.18 holds for filtered complexes endowed with  $\mathbb{Z}$ -indexed filtrations, because all basic results – i.e. Proposition 2.19, Proposition 2.20, Proposition 2.23 and Proposition 2.24 – extend to this generalised context with no major change in the proofs. However the description of cofibrant objects in the category of  $\mathbb{Z}$ -indexed filtered modules in complexes becomes even harder. Furthermore, in the geometric applications discussed in Chapter 2 we will only need bounded-below filtered complexes: this is the reason why we have chosen to stick to  $\mathbb{N}$ -indexed filtrations from the beginning.

Now assume  $R$  is a  $k$ -algebra, where  $k$  is a field of characteristic 0: we now endow  $\mathfrak{FdgMod}_R$  with the structure of a simplicially enriched category.

For all  $(M, F), (N, F) \in \mathfrak{FdgMod}_R$  consider the chain complex  $(\text{Hom}((M, F), (N, F)), \delta)$  defined as

$$\begin{aligned} \text{Hom}_{\mathfrak{FdgMod}_R}((M, F), (N, F))_n &:= \text{Hom}((M, F), (N[-n], F)) \\ \forall (f, F) \in \text{Hom}_{\mathfrak{FdgMod}_R}((M, F), (N, F))_n \quad \delta_n((f, F)) &\in \text{Hom}_{\mathfrak{FdgMod}_R}((M, F), (N, F))_{n-1} \\ \text{defined by } F^p(\delta_n((f, F))) &:= F^p \bar{d}^n \circ F^p f - (-1)^n F^p f \circ F^p d^n \end{aligned} \quad (2.13)$$

where, by a slight abuse of notation, we mean that  $F^p M[k] := (F^p M)[k]$ .

Formulae (2.13) make  $\mathfrak{FdgMod}_R$  into a differential graded category over  $k$ , so we can naturally endow it with a simplicial structure by taking denormalisation, i.e. by setting

$$\underline{\text{Hom}}_{\mathfrak{FdgMod}_R}((M, F), (N, F)) := \mathbf{K}(\tau_{\geq 0} \text{Hom}_{\mathfrak{FdgMod}_R}((M, F), (N, F))). \quad (2.14)$$

## 2.4 The Rees Functor

Let  $R$  be a commutative unital ring; the model structure over  $\mathfrak{FdgMod}_R$  given by Theorem 2.18 is really modelled on the unfiltered situation: unsurprisingly, the homotopy theories of filtered modules in complexes and unfiltered ones are closely related, and the functor connecting them is given by the classical Rees construction.

Recall that the *Rees module* associated to a filtered  $R$ -module  $(M, F)$  is defined to be the graded  $R[t]$ -module given by

$$\text{Rees}((M, F)) := \bigoplus_{p=0}^{\infty} F^p M \cdot t^{-p} \quad (2.15)$$

so the Rees construction transforms filtrations into grading with respect to the polynomial algebra  $R[t]$ . Also, it is quite evident from formula (2.15) that the Rees construction is functorial, so there is a functor<sup>6</sup>

$$\text{Rees} : \mathfrak{FMod}_R \longrightarrow \mathfrak{gMod}_{R[t]}$$

at our disposal, which in turn induces a functor<sup>7</sup>

$$\text{Rees} : \mathfrak{FdgMod}_R \longrightarrow \mathfrak{gdgMod}_{R[t]} \quad (2.16)$$

to the category of graded dg-modules over  $R[t]$ ; in particular we like to view the latter as the category  $\mathbb{G}_m\text{-dgMod}_{R[t]}$  of  $R[t]$ -modules in complexes equipped with an extra action of the multiplicative group compatible with the canonical action

$$\begin{aligned} \mathbb{G}_m \times \mathbb{A}_R^1 &\longrightarrow \mathbb{A}_R^1 \\ (\lambda, s) &\longmapsto \lambda^{-1}s \end{aligned} \quad (2.17)$$

The projective model structure on  $\mathfrak{dgMod}_{R[t]}$  admits a natural  $\mathbb{G}_m$ -equivariant version.

**Theorem 2.27** *Consider the sets*

$$\begin{aligned} I_{\mathbb{G}_m\text{-dgMod}_{R[t]}} &:= \{f : t^i S_{R[t]}(n+1) \rightarrow t^i D_{R[t]}(n)\}_{i,n \in \mathbb{Z}} \\ J_{\mathbb{G}_m\text{-dgMod}_{R[t]}} &:= \{f : 0 \rightarrow t^i D_{R[t]}(n)\}_{i,n \in \mathbb{Z}} \\ W_{\mathbb{G}_m\text{-dgMod}_{R[t]}} &:= \{f : M \rightarrow N \mid f \text{ is a } \mathbb{G}_m\text{-equivariant quasi-isomorphism}\}. \end{aligned} \quad (2.18)$$

<sup>6</sup>There is some abuse of notation in this formula.

<sup>7</sup>There is some abuse of notation in this formula.



The classes (2.18) determine a cofibrantly generated model structure over  $\mathbb{G}_m\text{-}\mathfrak{dgMod}_{R[t]}$ , in which  $I_{\mathbb{G}_m\text{-}\mathfrak{dgMod}_{R[t]}}$  is the set of generating cofibrations,  $J_{\mathbb{G}_m\text{-}\mathfrak{dgMod}_{R[t]}}$  is the set of generating trivial cofibrations and  $W_{\mathbb{G}_m\text{-}\mathfrak{dgMod}_{R[t]}}$  is the set of weak equivalences.

Arguments and lemmas discussed in Section 2.2 to prove Theorem 2.1 carry over to this context once we restrict to  $\mathbb{G}_m$ -equivariant objects and maps.

**Remark 2.28** Notice that maps in  $I_{\mathbb{G}_m\text{-}\mathfrak{dgMod}_{R[t]}}$  and  $J_{\mathbb{G}_m\text{-}\mathfrak{dgMod}_{R[t]}}$  are  $\mathbb{G}_m$ -equivariant, therefore all cofibrations are  $\mathbb{G}_m$ -equivariant.

Fibrations in the model structure determined by Theorem 2.18 are very nicely described: this is the content of the next proposition.

**Proposition 2.29**  $p \in \text{Hom}_{\mathbb{G}_m\text{-}\mathfrak{dgMod}_{R[t]}}(M, N)$  is a fibration if and only if it is a  $\mathbb{G}_m$ -equivariant degreewise surjection.

*Proof* The proof of Proposition 2.10 adapts to the  $\mathbb{G}_m$ -equivariant context.  $\square$

The following result collects various properties of functor (2.16): all claims are well-known, we only state them in homotopy-theoretical terms.

**Theorem 2.30** *The Rees functor*

$$\text{Rees} : \mathfrak{FdgMod}_R \longrightarrow \mathbb{G}_m\text{-}\mathfrak{dgMod}_{R[t]}.$$

has the following properties:

1. it has a left adjoint functor, given by

$$\begin{aligned} \varphi : \mathbb{G}_m\text{-}\mathfrak{dgMod}_{R[t]} &\longrightarrow \mathfrak{FdgMod}_R \\ M &\longmapsto (M_\varphi, F_\varphi) \\ M_\varphi &:= M/(1-t)M \\ F^n M_\varphi &:= \text{Im} (M^{\bullet, n} \rightarrow M_\varphi^\bullet) \end{aligned}$$

where the complex  $M$  is seen as a bigraded  $R[t]$ -module;

2. for all pairs  $(M, F)$ ,  $(N, F)$  there is bijection

$$\text{Hom}_R((M, F), (N, F)) \simeq \text{Hom}_{R[t]}(\text{Rees}((M, F)), \text{Rees}((N, F)))^{\mathbb{G}_m} \quad (2.19)$$

which is natural in all variables;

3. its essential image consists of the full subcategory of  $t$ -torsion-free  $R[t]$ -modules in complexes;
4. it induces an equivalence on the homotopy categories;
5. it preserves compact objects;<sup>9</sup>
6. it maps fibrations to fibrations.

<sup>8</sup>Here  $\text{Hom}_{R[t]}(\text{Rees}((M, F)), \text{Rees}((N, F)))^{\mathbb{G}_m}$  stands for the set of  $\mathbb{G}_m$ -equivariant morphisms of  $R[t]$ -modules in complexes between  $\text{Rees}((M, F))$  and  $\text{Rees}((N, F))$ .

<sup>9</sup>In particular this means that the Rees functor maps filtered perfect complexes to perfect complexes: we will be more precise about this in Sections 3.2 and 3.3.

In particular the Rees construction provides a Quillen equivalence between the categories  $\mathfrak{FdgMod}_R$  and  $\mathbb{G}_m\text{-}\mathfrak{dgmMod}_{R[t]}$ , both endowed with the projective model structure.

*Proof* We give references for most of the claims enunciated: the language we are using might be somehow different from the one therein, but the results and arguments we quote definitely apply to our statements.

1. Claim (1) follows from [13] Section 4.3: more specifically it is given by Comment 4.3.3;
2. Claim (2) follows from [28] Lemma 1.6;
3. Claim (3) follows from [13] Section 4.3: more specifically it is given again by Comment 4.3.3;
4. Claim (4) follows from [30] Theorem 3.16 and [30] Theorem 4.20: for a naiver explanation see [5] Section 3.1;
5. Claim (5) follows from [5] Section 3.1
6. In order to prove Claim (6), let  $f : (M, F) \rightarrow (N, F)$  be a fibration in  $\mathfrak{FdgMod}_R$ , i.e. by Proposition 2.19 assume that  $F^k f : F^k M \rightarrow F^k N$  is degreewise surjective for all  $k \in \mathbb{N}$ ; this in turn implies that

$$\begin{aligned} \text{Rees}(f) : \quad \text{Rees}((M, F)) &\rightarrow \quad \text{Rees}((N, F)) \\ \underline{x}^0 \oplus \underline{x}^1 \cdot t^{-1} \oplus \underline{x}^2 \cdot t^{-2} \oplus \cdots &\mapsto \underline{f}(x^0) \oplus F^1 f(\underline{x}^1) \cdot t^{-1} \oplus F^2 f(\underline{x}^2) \cdot t^{-2} \oplus \cdots \end{aligned}$$

is degreewise surjective as a map of  $\mathbb{G}_m$ -equivariant  $R[t]$ -modules in complexes, thus the statement follows because of Proposition 2.29.

In particular Claim (1), Claim (4) and Claim (6) can be rephrased by saying that

$$\text{Rees} : \mathfrak{FdgMod}_R \longrightarrow \mathbb{G}_m\text{-}\mathfrak{dgmMod}_{R[t]}.$$

is a right Quillen equivalence.  $\square$

*Remark 2.31* We can say that the model structure on  $\mathfrak{FdgMod}_R$  defined by Theorem 2.18 is precisely the one making the Rees functor into a right Quillen functor; more formally consider the pair given by the Rees functor and its left adjoint described in Theorem 2.30.1: than such a pair satisfies the assumption of Theorem 2.4 and moreover the model structure induced on  $\mathfrak{FdgMod}_R$  through the latter criterion is the one determined by Theorem 2.18.

*Remark 2.32* Relation (2.19) descends to Ext groups:  $\forall i \in \mathbb{Z}, \forall (M, F), (N, F) \in \mathfrak{FdgMod}_R$  we have that

$$\text{Ext}_R^i((M, F), (N, F)) \simeq \text{Ext}_{R[t]}^i(\text{Rees}((M, F)), \text{Rees}((N, F)))^{\mathbb{G}_m}$$

where the object on the left-hand side is the Ext group in the category  $\mathfrak{FdgMod}_R$ , i.e.

$$\text{Ext}_R^{n-i}((M, F), (N, F)) := \pi_i \underline{\text{Hom}}_{\mathfrak{FdgMod}_R}((M, F), (N[-n], F)).$$

### 3 Derived Moduli of Filtered Complexes

From now on  $k$  will always denote a field of characteristic 0 and  $R$  a (possibly differential graded) commutative algebra over  $k$ ; let  $X$  be a smooth proper scheme over  $k$ : the main goal of this chapter is to study derived moduli of filtered perfect complexes of  $\mathcal{O}_X$ -modules. In order to do this we will first recall some generalities about representability of derived stacks

– following the work of Lurie and Pridham – and then we will use these tools to construct derived geometric stacks classifying perfect complexes and filtered perfect complexes. Such stacks are related by a canonical forgetful map: as we will see in the last section of the chapter, the homotopy fibre of this map will help us define a coherent derived version of the Grassmannian.

### 3.1 Background on Derived Stacks and Representability

This section is devoted to collect some miscellaneous background material on derived geometric stacks which will be largely used in the other sections of this chapter: in particular we will review a few representability results – due to Lurie and Pridham – giving conditions for a simplicial presheaf on  $\mathrm{d}\mathcal{G}\mathcal{A}\mathcal{g}_R^{\leq 0}$  to give rise to a (truncated) derived geometric stack. We will assume that the reader is familiar with the notion of derived geometric  $n$ -stack and the basic tools of Derived Algebraic Geometry as they appear in the work of Lurie, Toën and Vezzosi: foundational references on this subject include [19, 20, 39, 41]; in any case throughout most of the paper it will be enough to think of a derived geometric stack as a functor  $\mathbf{F} : \mathrm{d}\mathcal{G}\mathcal{A}\mathcal{g}_R^{\leq 0} \rightarrow \mathcal{S}\mathcal{E}\mathcal{T}$  satisfying hyperdescent and some technical geometricity assumption – i.e. the existence of some sort of higher atlas – with respect to affine hypercovers. These two conditions are precisely those turning a completely abstract functor to some kind of “geometric space”, where the usual tools of Algebraic Geometry – such as quasi-coherent modules, formalism of the six operations, Intersection Theory – make sense. Also note that the case of derived schemes is much easier to figure out: as a matter of fact by [25] Theorem 6.42 a derived scheme  $\mathcal{X}$  over  $k$  can be seen as a pair  $(\pi^0\mathcal{X}, \mathcal{O}_{\mathcal{X},*})$ , where  $\pi^0\mathcal{X}$  is an honest  $k$ -scheme and  $\mathcal{O}_{\mathcal{X},*}$  is a presheaf of differential graded commutative algebras in non-positive degrees on the site of affine opens of  $\pi^0\mathcal{X}$  such that:

- the (cohomology) presheaf  $\mathcal{H}^0(\mathcal{O}_{\mathcal{X},*}) \simeq \mathcal{O}_{\pi^0\mathcal{X}}$ ;
- the (cohomology) presheaves  $\mathcal{H}^n(\mathcal{O}_{\mathcal{X},*})$  are quasi-coherent  $\mathcal{O}_{\pi^0\mathcal{X}}$ -modules.

**Warning 3.1** Be aware that there are some small differences between the definition of derived geometric stack given in [19] – which is the one we refer to in this paper – and the one given in [41]: for a comparison see the explanation provided in [25] and [39].

Now we are to recall representability for derived geometric stacks: all contents herein are adapted from [26] and [27].

Recall that a functor  $\mathbf{F} : \mathrm{d}\mathcal{G}\mathcal{A}\mathcal{g}_R^{\leq 0} \rightarrow \mathcal{S}\mathcal{E}\mathcal{T}$  is said to be *homotopic* or *homotopy-preserving* if it maps quasi-isomorphisms in  $\mathrm{d}\mathcal{G}\mathcal{A}\mathcal{g}_R^{\leq 0}$  to weak equivalences in  $\mathcal{S}\mathcal{E}\mathcal{T}$ , while it is called *homotopy-homogeneous* if for any morphism  $C \rightarrow B$  and any square-zero extension

$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0$$

in  $\mathrm{d}\mathcal{G}\mathcal{A}\mathcal{g}_R^{\leq 0}$  the natural map of simplicial sets<sup>10</sup>

$$\mathbf{F}(A \times_B C) \longrightarrow \mathbf{F}(A) \times_{\mathbf{F}(B)}^h \mathbf{F}(C)$$

is a weak equivalence.

<sup>10</sup>The symbol  $-\times_{-}^h-$  denotes the homotopy fibre product in  $\mathcal{S}\mathcal{E}\mathcal{T}$ .

Let  $\mathbf{F} : \mathfrak{dg}\mathfrak{Alg}_R^{\leq 0} \rightarrow \mathfrak{sSet}$  be a homotopy-preserving homotopy-homogeneous functor and take a point  $x \in \mathbf{F}(A)$ , where  $A \in \mathfrak{dg}\mathfrak{Alg}_R^{\leq 0}$ ; recall from [26] that the *tangent space* to  $\mathbf{F}$  at  $x$  is defined to be the functor

$$\begin{aligned} T_x \mathbf{F} : \mathfrak{dg}\mathfrak{Mod}_A^{\leq 0} &\longrightarrow \mathfrak{sSet} \\ M &\longmapsto \mathbf{F}(M \oplus A) \times_{\mathbf{F}(A)}^h \{x\} \end{aligned}$$

and define for any differential graded  $A$ -module  $M$  and for all  $i > 0$  the groups

$$D_x^{n-i}(\mathbf{F}, M) := \pi_i(T_x \mathbf{F}(M[-n])).$$

**Proposition 3.2** (Pridham) *In the notations of formula (3.1) we have that:*

1.  $\pi_i(T_x \mathbf{F}(M)) \simeq \pi_{i+1}(T_x \mathbf{F}(M[-1]))$ , so  $D_x^j(\mathbf{F}, M)$  is well-defined for all  $m$ ;
2.  $D_x^j(\mathbf{F}, M)$  is an abelian group and the abelian structure is natural in  $M$  and  $\mathbf{F}$ ;
3. there is a local coefficient system  $D^*(\mathbf{F}, M)$  on  $\mathbf{F}(A)$  whose stalk at  $x$  is  $D_x^*(\mathbf{F}, M)$ ;
4. for any map  $f : A \rightarrow B$  in  $\mathfrak{dg}\mathfrak{Alg}_R^{\leq 0}$  and any  $P \in \mathfrak{dg}\mathfrak{Mod}_B^{\leq 0}$  there is a natural isomorphism  $D_x^j(\mathbf{F}, f_* P) \simeq D_{f_* x}^j(\mathbf{F}, P)$ ;
5. let

$$0 \longrightarrow I \xrightarrow{e} A \xrightarrow{f} B \longrightarrow 0$$

be a square-zero extension in  $\mathfrak{dg}\mathfrak{Alg}_R^{\leq 0}$  and set  $y := f_* x$ : there is a long exact sequence of groups and sets

$$\begin{aligned} \cdots \xrightarrow{e_*} \pi_n(\mathbf{F}(A), x) &\xrightarrow{f_*} \pi_n(\mathbf{F}(B), y) \xrightarrow{o_e} D_x^{1-n}(\mathbf{F}, I) \xrightarrow{e_*} \pi_{n-1}(\mathbf{F}(A), x) \xrightarrow{f_*} \cdots \\ \cdots \xrightarrow{f_*} \pi_1(\mathbf{F}(B), y) &\xrightarrow{o_e} D_x^0(\mathbf{F}, I) \xrightarrow{-*x} \pi_0(\mathbf{F}(A)) \xrightarrow{f_*} \pi_0(\mathbf{F}(B)) \\ &\xrightarrow{o_e} \Gamma(\mathbf{F}(B), D^1(\mathbf{F}, I)). \end{aligned}$$

*Proof* Claim 1 and Claim 2 correspond to [26] Lemma 1.12, Claim 3 to [26] Lemma 1.16, Claim 4 to [26] Lemma 1.15 and Claim 5 to [26] Proposition 1.17.  $\square$

**Remark 3.3** Proposition 3.2 says that the sequence of abelian groups  $D_x^*(\mathbf{F}, M)$  should be thought morally as some sort of pointwise cohomology theory for the functor  $\mathbf{F}$ ; such a statement is actually true – in a rigorous mathematical sense – whenever  $\mathbf{F}$  is a derived geometric  $n$ -stack over  $R$  and  $x : \mathbb{R}\text{Spec}(A) \rightarrow \mathbf{F}$  is a point on it: as a matter of fact in this case

$$D_x^j(\mathbf{F}, M) = \text{Ext}_A^j(x^* \mathbb{L}^{F/R}, M).$$

At last, recall that a simplicial presheaf on  $\mathfrak{dg}\mathfrak{Alg}_R^{\leq 0}$  is said to be *nilcomplete* if for all  $A \in \mathfrak{dg}\mathfrak{Alg}_R^{\leq 0}$  the natural map

$$\mathbf{F}(A) \longrightarrow \varprojlim_r \mathbf{F}(P^r A)$$

is a weak equivalence, where  $\{P^r A\}_{r>0}$  stands for the Moore-Postnikov tower of  $A$  (see [7] for a definition).

Now we are ready to state *Lurie-Pridham Representability Theorem* for derived geometric stacks.

**Theorem 3.4** (Lurie, Pridham) *A functor  $\mathbf{F} : \mathfrak{dg}\mathfrak{Alg}_R^{\leq 0} \rightarrow \mathfrak{sSet}$  is a derived geometric  $n$ -stack almost of finite presentation if and only if the following conditions hold:*

1.  $\mathbf{F}$  is  $n$ -truncated;
2.  $\mathbf{F}$  is homotopy-preserving;
3.  $\mathbf{F}$  is homotopy-homogeneous;
4.  $\mathbf{F}$  is nilcomplete;
5.  $\pi^0 \mathbf{F}$  is a hypersheaf (for the étale topology);
6.  $\pi^0 \mathbf{F}$  preserves filtered colimits;
7. for finitely generated integral domains  $A \in H^0(R)$  and all  $x \in \mathbf{F}(A)$ , the groups  $D_x^j(\mathbf{F}, A)$  are finitely generated  $A$ -modules;
8. for finitely generated integral domains  $A \in H^0(R)$ , all  $x \in \mathbf{F}(A)$  and all étale morphisms  $f : A \rightarrow A'$ , the maps

$$D_x^*(\mathbf{F}, A) \otimes_A A' \longrightarrow D_{f_*x}^*(\mathbf{F}, A')$$

are isomorphisms;

9. for all finitely generated integral domains  $A \in \mathfrak{Alg}_{H^0(R)}$  and all  $x \in \mathbf{F}(A)$  the functors  $D_x^j(\mathbf{F}, -)$  preserve filtered colimits for all  $j > 0$ ;
10. for all complete discrete local Noetherian  $H^0(R)$ -algebras  $A$  the map

$$\mathbf{F}(A) \longrightarrow \varprojlim_r \mathbf{F}(A/\mathfrak{m}_A^r)$$

is a weak equivalence.

*Proof* See [26] Corollary 1.36 and lemmas therewith, which largely rely on [19] Theorem 7.5.1.  $\square$

**Remark 3.5** As we have already mentioned, a derived geometric  $n$ -stack roughly corresponds to a  $n$ -truncated homotopy-preserving simplicial presheaf on  $\mathfrak{dg}\mathfrak{Alg}_R^{\leq 0}$  which is a hypersheaf for the (homotopy) étale topology and which is obtained from an affine hypercover by taking successive smooth quotients. Theorem 3.4 says that in order to ensure that some given homotopy-homogeneous functor  $\mathbf{F} : \mathfrak{dg}\mathfrak{Alg}_R^{\leq 0} \rightarrow \mathfrak{sSet}$  is a derived geometric stack it suffices to verify that its underived truncation  $\pi^0 \mathbf{F} : \mathfrak{Alg}_{H^0(R)} \rightarrow \mathfrak{sSet}$  is a  $n$ -truncated stack (in the sense of [14] and [33]) and that for all  $x \in \mathbf{F}(A)$  the cohomology theories  $D_x^*(\mathbf{F}, -)$  satisfy some mild finiteness conditions.

The most technical assumption in Theorem 3.4 is probably Condition (4), i.e. nil-completeness: this is actually avoided when working with nilpotent algebras. Consider the full subcategory  $\mathfrak{dg}_b\mathfrak{Nil}_R^{\leq 0}$  of  $\mathfrak{dg}\mathfrak{Alg}_R^{\leq 0}$  made of bounded below differential graded commutative  $R$ -algebras in non-positive degrees such that the canonical map  $A \rightarrow H^0(A)$  is nilpotent: the following result is *Pridham Nilpotent Representability Criterion*.

**Theorem 3.6** (Pridham) *A functor  $\mathbf{F} : \mathfrak{dg}_b\mathfrak{Nil}_R^{\leq 0} \rightarrow \mathfrak{sSet}$  is the restriction of an almost finitely presented derived geometric  $n$ -stack  $\mathcal{F} : \mathfrak{dg}\mathfrak{Alg}_R^{\leq 0} \rightarrow \mathfrak{sSet}$  if and only if the following conditions hold:*

1.  $\mathbf{F}$  is  $n$ -truncated;

2.  $\mathbf{F}$  is homotopy-preserving;<sup>11</sup>
3.  $\mathbf{F}$  is homotopy-homogeneous;
4.  $\pi^0 \mathbf{F}$  is a hypersheaf (for the étale topology);
5.  $\pi^0 \mathbf{F}$  preserves filtered colimits;
6. for finitely generated integral domains  $A \in H^0(R)$  and all  $x \in \mathbf{F}(A)$ , the groups  $D_x^j(\mathbf{F}, A)$  are finitely generated  $A$ -modules;
7. for finitely generated integral domains  $A \in H^0(R)$ , all  $x \in \mathbf{F}(A)$  and all étale morphisms  $f : A \rightarrow A'$ , the maps

$$D_x^*(F, A) \otimes_A A' \longrightarrow D_{f_*x}^*(\mathbf{F}, A')$$

are isomorphisms;

8. for all finitely generated integral domains  $A \in \mathfrak{Alg}_{H^0(R)}$  and all  $x \in \mathbf{F}(A)$  the functors  $D_x^j(\mathbf{F}, -)$  preserve filtered colimits for all  $j > 0$ ;
9. for all complete discrete local Noetherian  $H^0(R)$ -algebras  $A$  the map

$$\mathbf{F}(A) \longrightarrow \varprojlim_r \mathbf{F}(A/\mathfrak{m}_A^r)$$

is a weak equivalence.

Moreover  $\mathcal{F}$  is uniquely determined by  $\mathbf{F}$  up to weak equivalence.

*Proof* See [26] Theorem 2.17. □

In the last part of this section we will recall from [27] a few criteria ensuring homotopicality, homogeneity and underived hyperdescent of a functor  $\mathbf{F} : \mathfrak{dgAlg}_R^{\leq 0} \rightarrow \mathfrak{sSet}$ , which from now on will always be thought of as an abstract derived moduli problem. Most definitions and results below will involve  $\mathfrak{sCat}$ -valued derived moduli functors rather than honest simplicial presheaves on  $\mathfrak{dgAlg}_R^{\leq 0}$ : the reason for this lies in the fact that it is often easier to tackle a derived moduli problem by considering a suitable  $\mathfrak{sCat}$ -valued functor  $\mathbf{F} : \mathfrak{dgAlg}_R^{\leq 0} \rightarrow \mathfrak{sCat}$  and then use Theorem 3.6 to prove that the diagonal of its simplicial nerve  $\text{diag}(\mathbf{BF}) : \mathfrak{dgAlg}_R^{\leq 0} \rightarrow \mathfrak{sSet}$  gives rise to a honest truncated derived geometric stack; we will see instances of such a procedure in Sections 3.2 and 3.3, for more examples see [27] Sections 4 and 5. Moreover Cegarra and Remedios showed in [3] that the diagonal of the simplicial nerve is weakly equivalent to the functor  $\tilde{W}$  obtained as the right adjoint of Illusie's total décalage functor (see [7] or [16] for a definition), so we can substitute  $\text{diag}(\mathbf{BF})$  with  $\tilde{W}\mathbf{F}$  in the above considerations: for more details see [27].

Let

$$\mathfrak{C} \xrightarrow{\mathbf{F}} \mathfrak{B} \xleftarrow{\mathbf{G}} \mathfrak{D}$$

be a diagram of simplicial categories; recall that the 2-fibre product  $\mathfrak{C} \times_{\mathfrak{B}}^{(2)} \mathfrak{D}$  is defined to be the simplicial category for which

$$\text{Ob}(\mathfrak{C} \times_{\mathfrak{B}}^{(2)} \mathfrak{D}) := \{(c, \theta, d) \mid c \in \mathfrak{C}, d \in \mathfrak{D}, \theta : \mathbf{F}(c) \rightarrow \mathbf{G}(d) \text{ is an isomorphism in } \mathfrak{B}_0\}$$

$$\text{Hom}_{\mathfrak{C} \times_{\mathfrak{B}}^{(2)} \mathfrak{D}}((c_1, \theta_1, d_1), (c_2, \theta_2, d_2)) := \{(f_1, f_2) \in \text{Hom}_{\mathfrak{C}} \times \text{Hom}_{\mathfrak{D}} \mid \mathbf{G}f_2 \circ \theta_1 = \theta_2 \circ \mathbf{F}f_1\}.$$

**Definition 3.7** A morphism  $\mathbf{F} : \mathfrak{C} \rightarrow \mathfrak{D}$  of simplicial categories is said to be a 2-fibration if the following conditions hold:

<sup>11</sup>When dealing with functors defined on  $\mathfrak{dgAlg}_R^{\leq 0}$  actually it suffices to check that tiny acyclic extensions are mapped to weak equivalences.

1.  $\forall c_1, c_2 \in \mathfrak{C}$ , the induced map  $\underline{\text{Hom}}_{\mathfrak{C}}(c_1, c_2) \rightarrow \underline{\text{Hom}}_{\mathfrak{D}}(\mathbf{F}(c_1), \mathbf{F}(c_2))$  is a fibration in  $\mathfrak{sSet}$ ;
2. for any  $c_1 \in \mathfrak{C}$ ,  $d \in \mathfrak{D}$  and homotopy equivalence  $h : \mathbf{F}(c_1) \rightarrow d$  in  $\mathfrak{C}$  there exist  $c_2 \in \mathfrak{C}$ , a homotopy equivalence  $k : c_1 \rightarrow c_2$  in  $\mathfrak{C}$  and an isomorphism  $\theta : \mathbf{F}(c_2) \rightarrow d$  such that  $\theta \circ \mathbf{F}k = h$ .

**Definition 3.8** A morphism  $\mathbf{F} : \mathfrak{C} \rightarrow \mathfrak{D}$  of simplicial categories is said to be a *trivial 2-fibration* if the following conditions hold:

1.  $\forall c_1, c_2 \in \mathfrak{C}$ , the induced map  $\underline{\text{Hom}}_{\mathfrak{C}}(c_1, c_2) \rightarrow \underline{\text{Hom}}_{\mathfrak{D}}(\mathbf{F}(c_1), \mathbf{F}(c_2))$  is a trivial fibration in  $\mathfrak{sSet}$ ;
2.  $\mathbf{F}_0 : \mathfrak{C}_0 \rightarrow \mathfrak{D}_0$  is essentially surjective.

**Definition 3.9** Fix two functors  $\mathbf{F}, \mathbf{G} : \mathfrak{dg}_b\mathfrak{Nil}_R^{\leq 0} \rightarrow \mathfrak{sCat}$ ; a natural transformation  $\eta : \mathbf{F} \rightarrow \mathbf{G}$  is said to be *2-homotopic* if for all tiny acyclic extensions  $A \rightarrow B$ , the natural map

$$\mathbf{F}(A) \longrightarrow \mathbf{F}(B) \times_{\mathbf{G}(B)}^{(2)} \mathbf{G}(A)$$

is a trivial 2-fibration. The functor  $\mathbf{F}$  is said to be 2-homotopic if so is the morphism  $\mathbf{F} \rightarrow \bullet$ .

**Definition 3.10** Fix two functors  $\mathbf{F}, \mathbf{G} : \mathfrak{dg}_b\mathfrak{Nil}_R^{\leq 0} \rightarrow \mathfrak{sCat}$ ; a natural transformation  $\eta : \mathbf{F} \rightarrow \mathbf{G}$  is said to be *formally 2-quasi-presmooth* if for all square-zero extensions  $A \rightarrow B$ , the natural map

$$\mathbf{F}(A) \longrightarrow \mathbf{F}(B) \times_{\mathbf{G}(B)}^{(2)} \mathbf{G}(A)$$

is a 2-fibration. If  $\eta$  is also 2-homotopic, it is said to be *formally 2-quasi-smooth*.

The functor  $\mathbf{F}$  is said to be *formally 2-quasi-(pre)smooth* if so is the morphism  $\mathbf{F} \rightarrow \bullet$ .

**Definition 3.11** A functor  $\mathbf{F} : \mathfrak{dg}_b\mathfrak{Nil}_R^{\leq 0} \rightarrow \mathfrak{sCat}$  is said to be *2-homogeneous* if for all square-zero extensions  $A \rightarrow B$  and all morphisms  $C \rightarrow B$  the natural map

$$\mathbf{F}(A \times_B C) \longrightarrow \mathbf{F}(A) \times_{\mathbf{F}(B)}^{(2)} \mathbf{F}(C)$$

is essentially surjective on objects and an isomorphism on  $\underline{\text{Hom}}$  spaces.

Now given a simplicial category  $\mathfrak{C}$ , denote by  $\mathcal{W}(\mathfrak{C})$  the maximal weak simplicial groupoid contained within  $\mathfrak{C}$ , i.e. the full simplicial subcategory of  $\mathfrak{C}$  in which morphisms are maps whose image in  $\pi_0\mathfrak{C}$  is invertible; in particular this means that  $\pi_0\mathcal{W}(\mathfrak{C})$  is the core of  $\pi_0\mathfrak{C}$ . Also denote by  $c(\pi_0\mathfrak{C})$  the set of isomorphism classes of the (honest) category  $\pi_0\mathfrak{C}$ .

The following result relates quasi-smoothness to homogeneity and will be very useful in the rest of the paper.

**Proposition 3.12** Let  $\mathbf{F} : \mathfrak{dg}_b\mathfrak{Nil}_R^{\leq 0} \rightarrow \mathfrak{sCat}$  be 2-homogeneous and formally 2-quasi-smooth; then

1.  $\text{diag}(\mathbf{BF})$  is homotopy-preserving;
2.  $\text{diag}(\mathbf{BF})$  is homotopy-homogeneous;
3. the map  $\mathcal{W}(\mathbf{F}) \rightarrow \mathbf{F}$  is formally étale, meaning that for any square-zero extension  $A \rightarrow B$  the induced map

$$\mathcal{W}(\mathbf{F}(A)) \longrightarrow \mathbf{F}(A) \times_{\mathbf{F}(B)} \mathcal{W}(\mathbf{F}(B))$$

is an isomorphism;

4.  $\mathcal{W}(\mathbf{F})$  is 2-homogeneous and formally 2-quasi smooth, as well.

*Proof* See [27], Section 3.3.  $\square$

At last, let us recall for future reference the notions of openness and (homotopy étaleness) for a  $\mathfrak{Set}$ -valued presheaf.

**Definition 3.13** Fix a presheaf  $\mathbf{C} : \mathcal{A}lg_{H^0(R)} \rightarrow \mathfrak{Set}$  and a subfunctor  $\mathbf{M} \subset \mathbf{C}$ ;  $\mathbf{M}$  is said to be a *functorial full simplicial subcategory* of  $\mathbf{C}$  if  $\forall A \in \mathcal{A}lg_{H^0(R)}, \forall X, Y \in \mathbf{M}(A)$ , the map

$$\underline{\mathrm{Hom}}_{\mathbf{M}(A)}(X, Y) \longrightarrow \underline{\mathrm{Hom}}_{\mathbf{C}(A)}(X, Y) \quad (3.1)$$

is a weak equivalence.

*Remark 3.14* In the notations of Definition 3.13, denote  $M := \bar{W}\mathcal{W}(\mathbf{M})$  and  $C := \bar{W}\mathcal{W}(\mathbf{C})$ ; formula (3.1) implies that the induced morphism  $M \rightarrow C$  is injective on  $\pi_0$  and bijective on all homotopy groups.

**Definition 3.15** Given a functor  $\mathbf{C} : \mathcal{A}lg_{H^0(R)} \rightarrow \mathfrak{Set}$  and a functorial simplicial subcategory  $\mathbf{M} \subset \mathbf{C}$ , say that  $\mathbf{M}$  is an *open simplicial subcategory* of  $\mathbf{C}$  if

1.  $\mathbf{M}$  is a full simplicial subcategory;
2. the map  $\mathbf{M} \rightarrow \mathbf{C}$  is *homotopy formally étale*, meaning that for any square-zero extension  $A \rightarrow B$ , the map

$$\pi_0 \mathbf{M}(A) \longrightarrow \pi_0 \mathbf{C}(A) \times_{\pi_0 \mathbf{C}(B)}^{(2)} \pi_0 \mathbf{M}(B)$$

is essentially surjective on objects.

**Proposition 3.16** (Pridham) Let  $\mathbf{C} : \mathcal{A}lg_{H^0(R)} \rightarrow \mathfrak{Set}$  be a functor for which

$$\bar{W}\mathcal{W}(\mathbf{C}) : \mathcal{A}lg_{H^0(R)} \rightarrow \mathfrak{Set}$$

is an étale hypersheaf and let  $\mathbf{M} \subset \mathbf{C}$  be functorial full simplicial subcategory. Then  $\bar{W}\mathcal{W}(\mathbf{M})$  is an étale hypersheaf if and only if for any  $A \in \mathcal{A}lg_{H^0(R)}$  and any étale cover  $\{f_\alpha : A \rightarrow B_\alpha\}_\alpha$  the map

$$c(\pi_0 \mathbf{M}(A)) \longrightarrow c(\pi_0 \mathbf{C}(A)) \times_{\prod_\alpha c(\pi_0 \mathbf{C}(B_\alpha))} \prod_\alpha c(\pi_0 \mathbf{M}(B_\alpha))$$

is surjective.

*Proof* See [27] Proposition 2.32.  $\square$

### 3.2 Derived Moduli of Perfect Complexes

Let  $X$  be a smooth proper scheme over  $k$  and recall that a complex  $\mathcal{E}$  of  $\mathcal{O}_X$ -modules is said to be *perfect* if it is compact as an object in the derived category  $D(X)$ ; in simpler terms  $\mathcal{E}$  is perfect if it is locally quasi-isomorphic to a bounded complex of vector bundles. A key example of perfect complex is given by the derived push-forward of the relative De Rham complex associated to a morphism of schemes: more clearly, if  $f : Y \rightarrow Z$  is a proper morphism of (semi-separated quasi-compact)  $k$ -schemes, then  $\mathbb{R}f_* \Omega_{Y/Z}$  is perfect as an object in  $D(Z)$ . Perfect complexes play a very important role in several parts of Algebraic Geometry – such as Hodge Theory, Deformation Theory, Enumerative Geometry, Symplectic Algebraic Geometry and Homological Mirror Symmetry – so it is very natural



to ask whether they can be classified by some moduli stack; for this reason consider the functor

$$\begin{aligned} \mathcal{P}erf_X^{\geq 0} : \mathcal{A}lg_k &\longrightarrow \mathfrak{G}rpd \\ A &\longmapsto \mathcal{P}erf_X^{\geq 0}(A) := \text{groupoid of perfect } (\mathcal{O}_X \otimes A)\text{-modules } \mathcal{E} \text{ in complexes} \\ &\quad \text{such that } \text{Ext}^i(\mathcal{E}, \mathcal{E}) = 0 \text{ for all } i < 0. \end{aligned} \quad (3.2)$$

**Theorem 3.17** (*Lieblich*) *Functor (3.2) is an (underived) Artin stack over  $k$  locally of finite presentation.*

*Proof* See [18] Theorem 4.2.1 and results therein.  $\square$

The assumptions on the base scheme  $X$  in Theorem 3.17 – whose proof relies on *Artin Representability Theorem* (see [1]) – can be relaxed, but the key condition of Lieblich’s result remains the vanishing of all negative Ext groups<sup>12</sup>; in particular observe that such a condition ensures that  $\mathcal{P}erf_X^{\geq 0}$  is a well-defined groupoid-valued functor: as a matter of fact the group  $\text{Ext}^i(\mathcal{E}, \mathcal{E})$ , where  $\mathcal{E} \in \mathbf{D}(X)$  and  $i < 0$ , parametrises  $i^{\text{th}}$ -order autoequivalences of  $\mathcal{E}$ , thus perfect complexes with trivial negative Ext groups do not carry any higher homotopy, but only usual automorphisms.

By means of Derived Algebraic Geometry it is possible to outstandingly generalise Lieblich’s result: indeed consider the functor

$$\begin{aligned} \mathbb{R}\mathcal{P}erf_X : \mathbf{dgAlg}_k^{\leq 0} &\longrightarrow \mathbf{sSet} \\ A &\longmapsto \text{Map}\left(\mathfrak{P}erf(X)^{\text{op}}, \hat{A}_{\text{pe}}\right) \end{aligned} \quad (3.3)$$

where  $\mathfrak{P}erf(X)$  stands for the dg-category of perfect complexes on  $X$ ,  $\hat{A}_{\text{pe}}$  for the dg-category of *perfect  $A$ -modules* (see [40] for more details) and  $\text{Map}$  for the mapping space of the model category of dg-categories (see [35, 37, 38] for more details).

**Theorem 3.18** (*Toën-Vaquié*) *Functor (3.3) is a locally geometric<sup>13</sup> derived stack over  $k$  locally of finite presentation.*

*Proof* See [40] Section 4; see also [39] Sections 3.2.4 and 4.3.5 for a quicker explanation.  $\square$

It is easily seen that there is a derived geometric 1-substack of  $\mathbb{R}\mathcal{P}erf_X$  whose underived truncation is equivalent to  $\mathcal{P}erf_X^{\geq 0}$ , so Theorem 3.17 is recovered as a corollary of Toën and Vaquié’s work.

Theorem 3.18 is a very powerful and elegant result, which has been highly inspiring in recent research: just to mention a few significant instances, it is one of the key ingredients in [34] where Simpson constructed a locally geometric stack of perfect complexes equipped with a  $\lambda$ -connection, [31] where Schürg, Toën and Vaquié constructed a derived determinant map from the derived stack of perfect complexes to the derived Picard stack and studied

<sup>12</sup>In [18] a perfect complex  $\mathcal{E} \in \mathbf{D}(X)$  such that  $\text{Ext}^i(\mathcal{E}, \mathcal{E}) = 0$  for all  $i < 0$  is called *universally gluable*; also in that paper the stack  $\mathcal{P}erf_X^{\geq 0}$  is denoted by  $\mathcal{D}_{\text{pug}}^b(X/k)$ .

<sup>13</sup>Recall that a derived stack  $\mathcal{F}$  is said to be *locally geometric* if it is the union of open truncated derived geometric substacks.

various applications to Deformation Theory and Enumerative Geometry, [24] where Pantev, Toën, Vaquié and Vezzosi set Derived Symplectic Geometry. However the proof provided in [40] is quite abstract and involved: as a matter of fact Toën and Vaquié actually constructed a derived stack parametrising *pseudo-perfect* objects (see [40] for a definition) in a fixed dg-category of finite type (again see [40] for more details) and then proved by hand – i.e. just by means of the definitions from [41], without any representability result – that this is locally geometric and locally of finite type. Theorem 3.18 is then obtained just as an interesting application.

In this section we will apply the representability and smoothness results discussed in Section 3.1 to obtain a new proof of Theorem 3.18; actually we will follow the path marked by Pridham in [27], where he develops general methods to study derived moduli of schemes and sheaves. In a way the approach we propose is the derived counterpart of Lieblich's one, as the latter is based on Artin Representability Theorem rather than the definition of (underived) Artin stack. Moreover we will give a rather explicit description of the derived geometric stacks determining the local geometricity of  $\mathbb{R}Perf_X$ : again, such a picture is certainly present in Toën and Vaquié's work, but unravelling the language in order to clearly write down the relevant substacks might be non-trivial. Halpern-Leistner and Preygel have recently studied the stack  $\mathbb{R}Perf_X$  via representability as well, though their approach does not make use of Pridham's theory: for more details see [9] Section 2.5. Other related work has been carried by Pandit, who showed in [23] that the derived moduli stack of compact objects in a perfect symmetric monoidal infinity-category is locally geometric and locally of finite type, and Lowrey, who studied in [17] the derived moduli stack of pseudo-coherent complexes on a proper scheme.

Let  $\mathcal{X}$  be a (possibly) derived scheme over  $R$  and recall that the *Cěch nerve* of  $\mathcal{X}$  associated to a fixed affine open cover  $U := \coprod_{\alpha} U_{\alpha}$  is defined to be the simplicial affine scheme

$$\check{\mathcal{X}} : \quad \check{\mathcal{X}}_0 \rightrightarrows \check{\mathcal{X}}_1 \rightrightarrows \check{\mathcal{X}}_2 \rightrightarrows \cdots$$

where

$$\check{\mathcal{X}}_m := \underbrace{U \times_X^h U \times_X^h \cdots \times_X^h U}_{m+1 \text{ times}}$$

while faces and degenerations are induced naturally by canonical projections and diagonal embeddings respectively. Consider also the cosimplicial differential graded commutative  $R$ -algebra  $\mathcal{O}(\mathcal{X})$  defined in level  $m$  by

$$\mathcal{O}(\mathcal{X})^m := \Gamma(\check{\mathcal{X}}_m, \mathcal{O}_{\check{\mathcal{X}}_m}) \quad (3.4)$$

whose cosimplicial structure maps are induced through the global section functor by the ones determining the simplicial structure of  $\check{\mathcal{X}}$ .

**Definition 3.19** Define a *derived module* over  $\mathcal{X}$  to be a cosimplicial  $\mathcal{O}(\mathcal{X})$ -module in complexes.

We will denote by  $\mathrm{dgMod}(\mathcal{X})$  the category of derived modules over  $\mathcal{X}$ ; just unravelling Definition 3.19 we see that an object  $\mathcal{M} \in \mathrm{dgMod}(\mathcal{X})$  is made of cochain complexes  $\mathcal{M}^m$  of  $\mathcal{O}(\mathcal{X})^m$ -modules related by maps

$$\begin{aligned} \partial^i : \mathcal{M}^m \otimes_{\mathcal{O}(\mathcal{X})^m}^{\mathbb{L}} \mathcal{O}(\mathcal{X})^{m+1} &\longrightarrow \mathcal{M}^{m+1} \\ \sigma^i : \mathcal{M}^m \otimes_{\mathcal{O}(\mathcal{X})^m}^{\mathbb{L}} \mathcal{O}(\mathcal{X})^{m-1} &\longrightarrow \mathcal{M}^{m-1} \end{aligned}$$

satisfying the usual cosimplicial identities. Observe that the projective model structures on cochain complexes we discussed in Section 2.2 induces a model structure on  $\mathfrak{dg}\mathfrak{Mod}(\mathcal{X})$ , which we will still refer to as a projective model structure: in particular a morphism  $f : \mathcal{M} \rightarrow \mathcal{N}$  in  $\mathfrak{dg}\mathfrak{Mod}(\mathcal{X})$  is

- a weak equivalence if  $f^m : \mathcal{M}^m \rightarrow \mathcal{N}^m$  is a quasi-isomorphism;
- a fibration if  $f^m : \mathcal{M}^m \rightarrow \mathcal{N}^m$  is degreewise surjective;
- a cofibration if it has the left lifting property with respect to all fibrations (see [27] Section 4.1 for a rather explicit characterisation of them).

In the same way, the category  $\mathfrak{dg}\mathfrak{Mod}(\mathcal{X})$  inherits a simplicial structure from the category of  $R$ -modules in complexes: more clearly for any  $\mathcal{M}, \mathcal{N} \in \mathfrak{dg}\mathfrak{Mod}(\mathcal{X})$  consider the chain complex  $(\mathrm{Hom}_{\mathcal{X}}(\mathcal{M}, \mathcal{N}), \delta)$  defined by the relations

$$\begin{aligned} \mathrm{Hom}_{\mathcal{X}}(\mathcal{M}, \mathcal{N})_n &:= \mathrm{Hom}_{O(\mathcal{X})}(\mathcal{M}, \mathcal{N}[-n]) \\ \forall f \in \mathrm{Hom}_{\mathcal{X}}(\mathcal{M}, \mathcal{N})_n \quad \delta_n(f) &:= \bar{d}^n \circ f - (-1)^n f \circ d^n \in \mathrm{Hom}_{\mathcal{X}}(\mathcal{M}, \mathcal{N})_{n-1} \end{aligned}$$

and define the Hom spaces just by taking good truncation and denormalisation, i.e. set

$$\underline{\mathrm{Hom}}_{\mathfrak{dg}\mathfrak{Mod}(\mathcal{X})}(\mathcal{M}, \mathcal{N}) := \mathbf{K}(\tau_{\geq 0} \mathrm{Hom}_{\mathcal{X}}(\mathcal{M}, \mathcal{N})).$$

**Definition 3.20** A *derived quasi-coherent sheaf* over  $\mathcal{X}$  is a derived module  $\mathcal{M}$  for which all face maps  $\partial^i$  are weak equivalences.

Let  $\mathfrak{dg}\mathfrak{Mod}(\mathcal{X})_{\mathrm{cart}}$  to be the full subcategory of  $\mathfrak{dg}\mathfrak{Mod}(\mathcal{X})$  consisting of derived quasi-coherent sheaves: this inherits a simplicial structure from the larger category and – even if it has not enough limits and thus cannot be a model category – it also inherits a reasonably well behaved subcategory of weak equivalences, so there is a homotopy category  $\mathrm{Ho}(\mathfrak{dg}\mathfrak{Mod}(\mathcal{X})_{\mathrm{cart}})$  of quasi-coherent modules over  $\mathcal{X}$  simply obtained by localising  $\mathfrak{dg}\mathfrak{Mod}(\mathcal{X})_{\mathrm{cart}}$  at weak equivalences.

**Remark 3.21** The constructions above make sense in a much wider generality: as a matter of fact in [27] Pridham defined derived quasi-coherent modules over any *homotopy derived Artin hypergroupoid* (see [25]) and through these objects he recovered the notion of homotopy-Cartesian module over a derived geometric stack which had previously been investigated by Toën and Vezzosi in [41]; also Corollary 3.23 – which is the main tool to deal with derived moduli of sheaves – holds in this much vaster generality. We have chosen to discuss derived quasi-coherent modules only for derived schemes since our goal is to study perfect complexes on a proper scheme, for which the full power of Pridham’s theory of Artin hypergroupoids is not really needed. In particular bear in mind that the Čech nerve of a derived scheme associated to an affine open cover is an example of homotopy Zariski 1-hypergroupoid.

From now on fix  $R$  to be an ordinary (underived)  $k$ -algebra and  $X$  to be a quasi-compact semi-separated scheme over  $R$ ; note that in [15] Hüttermann showed that

$$\mathrm{Ho}(\mathfrak{dg}\mathfrak{Mod}_{\mathrm{cart}}(X)) \simeq D(\mathfrak{L}\mathfrak{Coh}(X))$$

so in this case derived quasi-coherent modules are precisely what one would like them to be.

Now define the functor

$$\begin{aligned} d\mathrm{CART}_X : \mathfrak{dg}_b\mathfrak{Nil}_R^{\leq 0} &\longrightarrow \mathfrak{sCat} \\ A &\longmapsto (\mathfrak{dgMod}_{\mathrm{cart}}(X \otimes_R^{\mathbb{L}} A))^c \end{aligned} \quad (3.5)$$

where  $(\mathfrak{dgMod}_{\mathrm{cart}}(X \otimes_R^{\mathbb{L}} A))^c$  is the full simplicial subcategory of  $\mathfrak{dgMod}_{\mathrm{cart}}(X \otimes_R^{\mathbb{L}} A)$  on cofibrant objects, i.e. it is the (simplicial) category of cofibrant derived quasi-coherent modules on the derived scheme  $X \otimes_R^{\mathbb{L}} A$ .

**Proposition 3.22** (*Pridham*) *Functor (3.5) is 2-homogeneous and formally 2-quasi-smooth.*

*Proof* This is [27] Proposition 4.11, which relies on the arguments of [27] Proposition 3.7; we will discuss Pridham's proof here for the reader's convenience.

We first prove that  $d\mathrm{CART}_X$  is 2-homogeneous; let  $A \rightarrow B$  be a square-zero extension and  $C \rightarrow B$  a morphism in  $\mathfrak{dg}_b\mathfrak{Nil}_R^{\leq 0}$  and fix  $\mathcal{F}, \mathcal{F}' \in d\mathrm{CART}_X(A \times_B C)$ . Since by definition  $\mathcal{F}$  and  $\mathcal{F}'$  are cofibrant (i.e. degreewise projective by Proposition 2.12) we immediately have that the commutative square of simplicial sets

$$\begin{array}{ccc} \underline{\mathrm{Hom}}_{d\mathrm{CART}(A \times_B C)}(\mathcal{F}, \mathcal{F}') & \longrightarrow & \underline{\mathrm{Hom}}_{d\mathrm{CART}(A)}(\mathcal{F} \otimes_{A \times_B C} A, \mathcal{F}' \otimes_{A \times_B C} A) \\ \downarrow & & \downarrow \\ \underline{\mathrm{Hom}}_{d\mathrm{CART}(C)}(\mathcal{F} \otimes_{A \times_B C} C, \mathcal{F}' \otimes_{A \times_B C} C) & \longrightarrow & \underline{\mathrm{Hom}}_{d\mathrm{CART}(B)}(\mathcal{F} \otimes_{A \times_B C} B, \mathcal{F}' \otimes_{A \times_B C} B) \end{array}$$

is actually a Cartesian diagram. Moreover fix  $\mathcal{F}_A \in d\mathrm{CART}_X(A)$  and  $\mathcal{F}_C \in d\mathrm{CART}_X(C)$  and let  $\alpha : \mathcal{F}_A \otimes_A B \rightarrow \mathcal{F}_C \otimes_C B$  be an isomorphism; now define

$$\mathcal{F} := \mathcal{F}_A \otimes_{\alpha, \mathcal{F}_C \otimes_C B} \mathcal{F}_C \simeq \mathcal{F}_C \otimes_{\alpha, \mathcal{F}_A \otimes B} \mathcal{F}_A$$

which is a derived quasi-coherent module over  $X \otimes_R(A \otimes_B C)$ . Clearly we have that

$$\mathcal{F} \otimes_{A \times_B C} A \simeq \mathcal{F}_A \quad \mathcal{F} \otimes_{A \times_B C} C \simeq \mathcal{F}_C$$

and also observe that  $\mathcal{F}$  is cofibrant, i.e.  $\mathcal{F} \in d\mathrm{CART}_X(A \times_B C)$ . This shows that  $\underline{\mathrm{Hom}}_{d\mathrm{CART}_X}$  is homogeneous, which means that  $d\mathrm{CART}_X$  is a 2-homogeneous functor.

Now we prove that  $d\mathrm{CART}_X$  is formally 2-quasi-smooth; again let  $I \hookrightarrow A \twoheadrightarrow B$  be a square-zero extension and pick  $\mathcal{F}, \mathcal{F}' \in d\mathrm{CART}_X(A)$ . Observe that, since  $\mathcal{F}'$  is cofibrant as a quasi-coherent module over  $X \otimes_R^{\mathbb{L}} A$ , we have that the induced map  $\mathcal{F}' \rightarrow \mathcal{F}' \otimes_A B$  is still a square-zero extension; furthermore if  $A \rightarrow B$  is also a quasi-isomorphism, then so is  $\mathcal{F}' \rightarrow \mathcal{F}' \otimes_A B$ : as a matter of fact notice, as a consequence of Proposition 2.12, that

$$\ker(\mathcal{F}' \rightarrow \mathcal{F}' \otimes_A B) = \mathcal{F}' \otimes_A I.$$

Now it follows that the natural chain map

$$\mathrm{HOM}_{d\mathrm{CART}_X(A)}(\mathcal{F}, \mathcal{F}') \longrightarrow \mathrm{HOM}_{d\mathrm{CART}_X(B)}(\mathcal{F} \otimes_A B, \mathcal{F}' \otimes_A B)$$

is degreewise surjective and a quasi-isomorphism whenever so is  $A \rightarrow B$ . Now, by just applying truncation and Dold-Kan denormalisation, we get that the morphism of simplicial sets

$$\underline{\mathrm{Hom}}_{d\mathrm{CART}_X(A)}(\mathcal{F}, \mathcal{F}') \longrightarrow \underline{\mathrm{Hom}}_{d\mathrm{CART}_X(B)}(\mathcal{F} \otimes_A B, \mathcal{F}' \otimes_A B)$$

is a fibration, which is trivial in case the square-zero extension  $A \rightarrow B$  is a quasi-isomorphism. This shows that  $\underline{\mathrm{Hom}}_{d\mathrm{CART}_X}$  is formally quasi-smooth, so in order to finish the proof we only need to prove that the base-change morphism

$$d\mathrm{CART}_X(A) \longrightarrow d\mathrm{CART}_X(B) \quad (3.6)$$

is a 2-fibration, which is trivial whenever the extension  $A \rightarrow B$  is acyclic. The computations in [25] Section 7 imply that obstructions to lifting a quasi-coherent module  $\mathcal{F} \in d\mathrm{CART}_X(B)$  to  $d\mathrm{CART}_X(A)$  lie in the group

$$\mathrm{Ext}_{X \otimes_R^{\mathbb{L}} B}^2(\mathcal{F}, \mathcal{F} \otimes_B I).$$

so in particular if  $H^*(I) = 0$  then map (3.6) is a trivial 2-fibration. Now fix  $\mathcal{F} \in d\mathrm{CART}_X(A)$ , denote  $\tilde{\mathcal{F}} := \mathcal{F} \otimes_A B$  and let  $\theta : \tilde{\mathcal{F}} \rightarrow \mathcal{G}$  be a homotopy equivalence in  $d\mathrm{CART}_X(B)$ . By cofibrancy, there exist a unique lift  $\mathcal{G}$  of  $\mathcal{G}$  to  $A$  as a cosimplicial graded module and, in the same fashion, we can lift  $\theta$  to a graded morphism  $\tilde{\theta} : \mathcal{F} \rightarrow \mathcal{G}$ : we want to prove that there also exist compatible lifts of the differential. The obstruction to lift the differential  $d$  of  $\mathcal{G}$  to a differential  $\delta$  on  $\mathcal{G}$  is given by a pair

$$(u, v) \in \mathrm{HOM}_{X \otimes_R^{\mathbb{L}} B}^2(\mathcal{G}, \mathcal{G} \otimes_B I) \times \mathrm{HOM}_{X \otimes_R^{\mathbb{L}} B}^1(\tilde{\mathcal{F}}, \mathcal{G} \otimes_B I)$$

satisfying  $d(u) = 0$  and  $d(v) = u \circ \theta$ . A different choice for  $(\delta, \tilde{\theta})$  would be of the form  $(\delta + a, \tilde{\theta} + b)$ , with

$$(a, b) \in \mathrm{HOM}_{X \otimes_R^{\mathbb{L}} B}^1(\mathcal{G}, \mathcal{G} \otimes_B I) \times \mathrm{HOM}_{X \otimes_R^{\mathbb{L}} B}^0(\tilde{\mathcal{F}}, \mathcal{G} \otimes_B I)$$

so that the pair  $(u, v)$  is sent to  $(u + d(a), v + d(b) + a \circ \theta)$ . It follows that the obstruction to lifting  $\theta$  and  $\mathcal{G}$  lies in

$$H^2\left(\mathrm{cone}\left(\mathrm{HOM}_{X \otimes_R^{\mathbb{L}} B}(\mathcal{G}, \mathcal{G} \otimes_B I) \xrightarrow{\theta^*} \mathrm{HOM}_{X \otimes_R^{\mathbb{L}} B}(\tilde{\mathcal{F}}, \mathcal{G} \otimes_B I)\right)\right). \quad (3.7)$$

Since  $\theta$  is a homotopy equivalence we have that  $\theta^*$  is a quasi-isomorphism: in particular the cohomology group (3.7) is 0, which means that suitable lifts exist. This completes the proof.  $\square$

Proposition 3.22 is the key ingredient to build upon Pridham Nilpotent Representability Criterion an ad-hoc result to deal with moduli of sheaves.

**Corollary 3.23** (Pridham) *Let*

$$\mathbf{M} : \mathcal{A}lg_R \rightarrow \mathcal{sCat}$$

*be a presheaf satisfying the following conditions:*

1.  $\mathbf{M}$  is  $n$ -truncated;
2.  $\mathbf{M}$  is open in the functor

$$A \longmapsto \pi^0 \mathcal{W}\left(\mathrm{d}g\mathrm{Mod}\left(X \otimes_R^{\mathbb{L}} A\right)_{\mathrm{cart}}\right) \quad A \in \mathcal{A}lg_R;$$

3. *If  $\{f_\alpha : A \rightarrow B_\alpha\}_\alpha$  is an étale cover in  $\mathcal{A}lg_R$ , then  $\mathcal{E} \in \pi_0 \pi^0 \mathcal{W}\left(\mathrm{d}g\mathrm{Mod}\left(X \otimes_R^{\mathbb{L}} A\right)_{\mathrm{cart}}\right)$  lies in the essential image of  $\pi_0 \mathbf{M}(A)$  whenever  $(f_\alpha)^* \mathcal{E}$  is in the essential image of  $\pi_0 \mathbf{M}(B_\alpha)$  for all  $\alpha$ ;*

4. For all finitely generated  $A \in \mathfrak{Alg}_R$  and all  $\mathcal{E} \in \mathbf{M}(A)$ , the functors

$$\mathrm{Ext}_{X \otimes_R^{\mathbb{L}} A}^i(\mathcal{E}, \mathcal{E} \otimes_A^{\mathbb{L}} -) : \mathfrak{Mod}_A \longrightarrow \mathfrak{Ab}$$

preserve filtered colimits  $\forall i \neq 1$ ;

5. For all finitely generated integral domains  $A \in \mathfrak{Alg}_R$  and all  $\mathcal{E} \in \mathbf{M}(A)$ , the groups  $\mathrm{Ext}_{X \otimes_R^{\mathbb{L}} A}^i(\mathcal{E}, \mathcal{E})$  are finitely generated  $A$ -modules;

6. The functor

$$c(\pi_0 \mathbf{M}) : \mathfrak{Alg}_R \longrightarrow \mathfrak{Set}$$

of components of the groupoid  $\pi_0 \mathbf{M}$  preserves filtered colimits;

7. For all complete discrete local Noetherian normal  $R$ -algebras  $A$ , for all  $\mathcal{E} \in \mathbf{M}(A)$  and for all  $i > 0$  the canonical maps

$$c(\pi_0 \mathbf{M}(A)) \longrightarrow \varprojlim_r c(\pi_0 \mathbf{M}(A/\mathfrak{m}_A^r))$$

$$\mathrm{Ext}_{X \otimes_R^{\mathbb{L}} A}^i(\mathcal{E}, \mathcal{E}) \longrightarrow \varprojlim_r \mathrm{Ext}_{X \otimes_R^{\mathbb{L}} A}^i(\mathcal{E}, \mathcal{E}/\mathfrak{m}_A^r) \quad \forall i < 0$$

are isomorphisms.

Let

$$\check{\mathbf{M}} : \mathfrak{dg}_b \mathfrak{Nil}_R^{\leq 0} \longrightarrow \mathfrak{sCat}$$

be the full simplicial subcategory of  $\mathcal{W}(d\mathrm{CART}_X(A))$  consisting of objects  $\mathcal{F}$  such that the complex  $\mathcal{F} \otimes_A H^0(A)$  is weakly equivalent in  $\mathfrak{dgMod}_{\mathrm{cart}}(X \otimes_R^{\mathbb{L}} H^0(A))$  to an object of  $\mathbf{M}(H^0(A))$ . Then the functor  $\check{\mathbf{W}}\check{\mathbf{M}}$  is (the restriction to  $\mathfrak{dg}_b \mathfrak{Nil}_R^{\leq 0}$  of) a derived geometric  $n$ -stack.

*Proof* This is [27] Theorem 4.12; we just sketch the main ideas of the proof for the reader's convenience. We basically need to verify that the various conditions in the statement imply that the simplicial presheaf  $\check{\mathbf{W}}\check{\mathbf{M}}$  satisfies Pridham Nilpotent Representability Criterion (Theorem 3.6).

First observe that by Condition (2) we have that

$$\check{\mathbf{M}}(A) \approx \mathbf{M}(H^0(A)) \times_{\mathcal{W}(\mathfrak{dgMod}(X \otimes_R^{\mathbb{L}} H^0(A))_{\mathrm{cart}})}^h \mathcal{W}(\mathfrak{dgMod}(X \otimes_R^{\mathbb{L}} A)_{\mathrm{cart}}).^{14} \quad (3.8)$$

As a matter of fact, the openness of  $\mathbf{M}$  inside  $\pi^0 \mathcal{W}(\mathfrak{dgMod}(X \otimes_R^{\mathbb{L}} -)_{\mathrm{cart}})$  says that the inclusion

$$\mathbf{M} \hookrightarrow \pi^0 \mathcal{W}(\mathfrak{dgMod}(X \otimes_R^{\mathbb{L}} -)_{\mathrm{cart}})$$

is homotopy formally étale, thus we have the representation given by formula (3.8).

Then note that the proof of [25] Lemma 5.23 adapts to  $\mathcal{O}_X$ -modules, i.e. the assignment

$$A \longmapsto \mathfrak{dgMod}_{X \otimes_R^{\mathbb{L}} A}$$

provides us with a left Quillen hypersheaf, thus [25] Proposition 5.9 implies that  $\check{\mathbf{W}}\mathcal{W}(d\mathrm{CART}_X)$  is an étale hypersheaf; now Condition (3) and Proposition 3.16 ensure that  $\check{\mathbf{W}}\mathcal{W}(\mathbf{M})$  is a hypersheaf for the étale topology.

<sup>14</sup>The symbol  $\approx$  stands for “weakly equivalent”.

Also recall that the computations in [25] Section 7 imply that

$$D_{\mathcal{E}}^i(\bar{W}\check{M}, M) \simeq \text{Ext}_{X \otimes_A^{\mathbb{L}} M}^{i+1}(\mathcal{E}, \mathcal{E} \otimes_A^{\mathbb{L}} M) \quad (3.9)$$

for all nilpotent cda's  $A \in \mathfrak{d}\mathfrak{g}_b\mathfrak{N}\mathfrak{il}_R^{\leq 0}$ , all complexes  $\mathcal{E} \in \bar{W}\check{M}(A)$  and dg  $A$ -modules  $M$ .

Now Proposition 3.22 and Proposition 3.12 tell us that Condition (4) and Condition (5) imply the homotopy-theoretic properties required by Pridham Nilpotent Representability Criterion, while the description of cohomology theories given by (3.9) ensures the compatibility of such modules with filtered colimits and base-change. In the end the weak completeness condition given by Condition (9) of Theorem 3.6 follows from Condition (7) through a few standard Mittag-Leffler computations: for more details see [27] Theorem 4.12 or the proof of Theorem 3.24, where similar calculations will be explicitly developed.  $\square$

Now we are ready to study derived moduli of perfect complexes by means of Lurie-Pridham representability; consider the functor

$$\begin{aligned} \mathbf{M}^n : \mathfrak{Alg}_R &\longrightarrow \mathfrak{Set} \\ A &\longmapsto \mathbf{M}^n(A) := \text{full simplicial subcategory} \\ &\quad \text{of perfect complexes } \mathcal{E} \text{ of } (\mathcal{O}_X \otimes_R^{\mathbb{L}} A)\text{-modules} \\ &\quad \text{such that } \text{Ext}_{X \otimes_R^{\mathbb{L}} A}^i(\mathcal{E}, \mathcal{E}) = 0 \text{ for } i < -n \end{aligned} \quad (3.10)$$

which classifies perfect  $\mathcal{O}_X$ -modules in complexes with trivial Ext groups in higher negative degrees.

**Theorem 3.24** *In the above notations, assume that the scheme  $X$  is smooth and proper; then functor (3.10) induces a derived geometric  $n$ -stack  $\mathbb{R}\text{Per} f_X^n$ .*

*Proof* We have to prove that functor (3.10) satisfies the conditions of Corollary 3.23.

First of all, notice that the vanishing condition on higher negative Ext groups guarantees that the simplicial presheaf  $\mathbf{M}^n$  is  $n$ -truncated, which is exactly Condition (1).

Now we look at Condition (2), hence we need to prove the openness of  $\mathbf{M}^n$  as a subfunctor of  $\pi^0\mathcal{W}(\mathfrak{d}\mathfrak{g}\mathfrak{M}\mathfrak{od}(X \otimes_R^{\mathbb{L}} -)_{\text{cart}})$ ; it is immediate to see that  $\mathbf{M}^n(A)$  is a full simplicial subcategory of  $\pi^0\mathcal{W}(\mathfrak{d}\mathfrak{g}\mathfrak{M}\mathfrak{od}(X \otimes_R^{\mathbb{L}} A)_{\text{cart}})$ , so we only need to check that the map

$$\mathbf{M}^n \hookrightarrow \pi^0\mathcal{W}(\mathfrak{d}\mathfrak{g}\mathfrak{M}\mathfrak{od}(X \otimes_R^{\mathbb{L}} -)_{\text{cart}}) \quad (3.11)$$

is homotopy formally étale, i.e. that the morphism of formal groupoids<sup>15</sup>

$$\pi_0\mathbf{M}^n \hookrightarrow \pi_0\pi^0\mathcal{W}(\mathfrak{d}\mathfrak{g}\mathfrak{M}\mathfrak{od}(X \otimes_R^{\mathbb{L}} -)_{\text{cart}})$$

is formally étale. By classical Formal Deformation Theory this amounts to check that the map induced by morphism (3.11) is an isomorphism on tangent spaces and an injection on obstruction spaces (see for example [32] Section 3.1 and [22] Section V.8), so fix a square-zero extension  $I \hookrightarrow A \twoheadrightarrow B$  and a perfect complex  $\mathcal{E} \in \mathbf{M}^n(B)$ . By Lieblich's work (see [18] Section 4) we have that

- the tangent space to the functor  $\pi_0\mathbf{M}^n$  at  $\mathcal{E}$  is given by the group  $\text{Ext}_{X \otimes_R^{\mathbb{L}} A}^1(\mathcal{E}, \mathcal{E} \otimes_B^{\mathbb{L}} I)$ ;

<sup>15</sup>Notice that (homotopy) formal étaleness is a local property, so we can restrict map (3.11) to formal objects.

- a functorial obstruction space for  $\pi_0 \mathbf{M}^n$  at  $\mathcal{E}$  is given by the group  $\mathrm{Ext}_{X \otimes_R^{\mathbb{L}} A}^2(\mathcal{E}, \mathcal{E} \otimes_B^{\mathbb{L}} I)$ .

On the other hand, it is well known (for instance see the proof of [27] Theorem 4.12) that

- the tangent space to the functor  $\pi_0 \pi^0 \mathcal{W}(\mathrm{d}\mathfrak{g}\mathfrak{M}\mathrm{od}(X \otimes_R^{\mathbb{L}} -)_{\mathrm{cart}})$  at  $\mathcal{E}$  is given by the group  $\mathrm{Ext}_{X \otimes_R^{\mathbb{L}} A}^1(\mathcal{E}, \mathcal{E} \otimes_B^{\mathbb{L}} I)$ ;
- a functorial obstruction space for  $\pi_0 \pi^0 \mathcal{W}(\mathrm{d}\mathfrak{g}\mathfrak{M}\mathrm{od}(X \otimes_R^{\mathbb{L}} -)_{\mathrm{cart}})$  at  $\mathcal{E}$  is given by the group  $\mathrm{Ext}_{X \otimes_R^{\mathbb{L}} A}^2(\mathcal{E}, \mathcal{E} \otimes_B^{\mathbb{L}} I)$ .

It follows that the group homomorphism induced by map (3.11) on first-order deformations and obstruction theories is just the identity, so Condition (2) holds.

Now let us look at Condition (3): take an étale cover  $\{f_\alpha : A \rightarrow B_\alpha\}_\alpha$  in  $\mathfrak{Alg}_R$  and let  $\mathcal{E}$  be an object in  $\pi_0 \pi^0 \mathcal{W}(\mathrm{d}\mathfrak{g}\mathfrak{M}\mathrm{od}(X \otimes_R^{\mathbb{L}} A)_{\mathrm{cart}})$  such that the derived modules  $(f_\alpha)^* \mathcal{E}$  over  $X \otimes_R^{\mathbb{L}} B_\alpha$  are perfect; then the derived quasi-coherent module  $\mathcal{E}$  has to be perfect as well, because perfectness is a local property which is preserved under pull-back. It follows that Condition (3) holds.

In order to check Condition (4), fix a finitely generated  $R$ -algebra  $A$  and a perfect complex  $\mathcal{E}$  of  $(\mathcal{O}_X \otimes_R^{\mathbb{L}} A)$ -modules and consider an inductive system  $\{B_\alpha\}_\alpha$  of  $A$ -algebras. The perfectness assumption on  $\mathcal{E}$  allows us to substitute this with a bounded complex  $\mathcal{F}$  of flat  $(\mathcal{O}_X \otimes_R^{\mathbb{L}} A)$ -modules, so we get that  $\mathrm{Ext}_{X \otimes_R^{\mathbb{L}} A}^i(\mathcal{E}, \mathcal{E} \otimes_A^{\mathbb{L}} -)$  preserves filtered colimits if and only if so does  $\mathrm{Ext}_{X \otimes_R^{\mathbb{L}} A}^i(\mathcal{E}, \mathcal{F} \otimes_A -)$ , which is just the classical Ext functor. Now a few standard results in Homological Algebra imply the following canonical isomorphisms  $\forall i \geq 0$

$$\begin{aligned} \mathrm{Ext}_{X \otimes_R^{\mathbb{L}} A}^i\left(\mathcal{E}, \mathcal{F} \otimes_A \varinjlim_\alpha B_\alpha\right) &\simeq \mathrm{Ext}_{X \otimes_R^{\mathbb{L}} A}^i\left(\mathcal{E}, \varinjlim_\alpha \mathcal{F} \otimes_A B_\alpha\right) \\ &\simeq \varinjlim_\alpha \mathrm{Ext}_{X \otimes_R^{\mathbb{L}} A}^i(\mathcal{E}, \mathcal{F} \otimes_A B_\alpha). \end{aligned}$$

In particular in the first isomorphism we are using the fact that filtered colimits commute with exact functors (and so is the tensor product as  $\mathcal{F}$  is flat in each degree), while in the second one we are using the fact that filtered colimits commute with all Ext functors, since  $\mathcal{E}$  is a finitely presented object as by perfectness this is locally quasi-isomorphic to a bounded complex of vector bundles. Ultimately the key idea in this argument is that the assumptions on the complexes we are classifying allow us to compute the Ext groups by choosing a “projective resolution” in the first variable and a “flat resolution” in the second one, so that all necessary finiteness conditions to make  $\mathrm{Ext}_{X \otimes_R^{\mathbb{L}} A}^i$  and  $\varinjlim_\alpha$  commute are

verified (see [42] Section 2.6). It follows that Condition (4) holds.

In order to check Condition (5), fix a finitely generated  $R$ -algebra  $A$  and a perfect complex  $\mathcal{E}$  of  $(\mathcal{O}_X \otimes_R^{\mathbb{L}} A)$ -modules and again choose  $\mathcal{F}$  to be a bounded complex of flat  $(\mathcal{O}_X \otimes_R^{\mathbb{L}} A)$ -modules being quasi-isomorphic to  $\mathcal{E}$ . Consider the derived endomorphism complex of  $\mathcal{E}$  over  $X \otimes_R^{\mathbb{L}} A$ : we have that

$$\mathbb{R}\mathrm{Hom}_{\mathcal{O}_X \otimes_R^{\mathbb{L}} A}(\mathcal{E}, \mathcal{E}) \approx (\mathcal{E})^\vee \otimes_{\mathcal{O}_X \otimes_R^{\mathbb{L}} A} \mathcal{F}$$

where

$$\mathcal{E}^\vee := \mathbb{R}\mathrm{Hom}_{\mathcal{O}_X \otimes_R^{\mathbb{L}} A}(\mathcal{E}, \mathcal{O}_X).$$



Notice that, again, we have computed the complex  $\mathbb{R}\mathcal{H}om_{\mathcal{O}_X \otimes_R^{\mathbb{L}} A}(\mathcal{E}, \mathcal{E})$  by choosing a “flat resolution” in the second entry and a “projective resolution” in the first one; now consider the cohomology sheaves

$$\mathcal{E}xt_{\mathcal{O}_X \otimes_R^{\mathbb{L}} A}^i(\mathcal{E}, \mathcal{E}) := \mathcal{H}^i\left(\mathbb{R}\mathcal{H}om_{\mathcal{O}_X \otimes_R^{\mathbb{L}} A}(\mathcal{E}, \mathcal{E})\right)$$

and note that these are coherent  $\mathcal{O}_X \otimes_R^{\mathbb{L}} A$ -modules. The local-to-global spectral sequence

$$H^p\left(X, \mathcal{E}xt_{\mathcal{O}_X \otimes_R^{\mathbb{L}} A}^q(\mathcal{E}, \mathcal{E})\right) \Longrightarrow \mathrm{Ext}_{X \otimes_R^{\mathbb{L}} A}^{p+q}(\mathcal{E}, \mathcal{E}) \quad (3.12)$$

relates the cohomology of the  $\mathcal{E}xt$  sheaves to the  $\mathrm{Ext}$  groups and is well-known to converge: since the sheaves  $\mathcal{E}xt_{\mathcal{O}_X \otimes_R^{\mathbb{L}} A}^i(\mathcal{E}, \mathcal{E})$  are coherent and finitely many, formula (3.12) implies that the groups  $\mathrm{Ext}_{X \otimes_R^{\mathbb{L}} A}^{p+q}(\mathcal{E}, \mathcal{E})$  are finitely generated as  $A$ -modules, thus Condition (5) holds.

Now we look at Condition (6); fix an inductive system  $\{A_\alpha\}_\alpha$  of  $R$ -algebras and let  $A := \varinjlim_\alpha A_\alpha$ : we need to show that

$$c\left(\pi_0 \mathbf{M}^n(A)\right) = \varinjlim_\alpha c\left(\pi_0 \mathbf{M}^n(A_\alpha)\right) \quad (3.13)$$

where for any  $R$ -algebra  $B$

$$c\left(\pi_0 \mathbf{M}^n(B)\right) := \left\{ \text{isomorphism classes of perfect complexes of } \left(\mathcal{O}_X \otimes_R^{\mathbb{L}} B\right)\text{-modules} \right\}.$$

Because being a perfect complex is local property, it suffices to show that formula (3.13) holds locally, i.e replacing  $X$  with an open affine subscheme  $U$ ; in particular, as flat modules are locally free, observe that a class  $[M] \in c\left(\pi_0 \mathbf{M}^n(B)\right)$  is locally determined by an equivalence class of bounded complexes

$$M_1 \xrightarrow{d} M_2 \xrightarrow{d} \cdots \xrightarrow{d} M_s \quad (3.14)$$

where  $s$  is some natural number and  $M_i$  is a free  $\mathcal{O}_X(U) \otimes_R^{\mathbb{L}} B$ -module for all  $i$ ; again we have used the property that perfect complexes are quasi-isomorphic to bounded and degreewise flat ones. Now denote by  $i_k$  the rank of the module  $M_k$  in representative (3.14) and consider the scheme defined for all  $B \in \mathfrak{Alg}_R$  through the functor of points

$$S(B) := \left\{ (D_i) \in \prod_{k=1, \dots, s-1} \mathrm{Mat}_{i_k, i_{k+1}}\left(\mathcal{O}_X(U) \otimes_R^{\mathbb{L}} B\right) \text{ s.t. } D_i^2 = 0 \right\}. \quad (3.15)$$

Formula (3.15) determines a closed subscheme of

$$\prod_{k=1, \dots, s-1} \mathrm{Mat}_{i_k, i_{k+1}}\left(\mathcal{O}_X(U) \otimes_R^{\mathbb{L}} B\right)$$

and provides a local description of  $c\left(\pi_0 \mathbf{M}^n(B)\right)$ ; clearly

$$\prod_{k=1, \dots, s-1} \mathrm{Mat}_{i_k, i_{k+1}}\left(\mathcal{O}_X(U) \otimes_R^{\mathbb{L}} A\right) \simeq \varinjlim_\alpha \prod_{k=1, \dots, s-1} \mathrm{Mat}_{i_k, i_{k+1}}\left(\mathcal{O}_X(U) \otimes_R^{\mathbb{L}} A_\alpha\right) \quad (3.16)$$

and since the subscheme  $S \hookrightarrow \prod_{k=1, \dots, s-1} \text{Mat}_{i_k, i_{k+1}}$  is defined by finitely many equations, formula (3.16) descends to  $S(A)$ , meaning that

$$S(A) \simeq \varinjlim_{\alpha} S(A_{\alpha}). \quad (3.17)$$

Formula (3.17) implies formula (3.13), so Condition (6) holds.

Lastly, we have to check Condition (7), so fix a complete discrete local Noetherian  $R$ -algebra  $A$  and a perfect complex  $\mathcal{E}$  of  $\mathcal{O}_X \otimes_R^{\mathbb{L}} A$ ; again the assumptions on  $\mathcal{E}$  allow us to substitute it with a bounded complex  $\mathcal{F}$  of flat  $\mathcal{O}_X \otimes_R^{\mathbb{L}} A$ -modules.

We first prove the compatibility of the Ext functors; the properties of  $A$  imply that the canonical morphism  $A \rightarrow \hat{A}$  to the pro-nilpotent completion

$$\hat{A} := \varprojlim_r A/m_A^r \quad (3.18)$$

is an isomorphism, which we can use to induce  $\forall i > 0$  a canonical isomorphism

$$\text{Ext}_{X \otimes_R^{\mathbb{L}} A}^i(\mathcal{E}, \mathcal{F}) \xrightarrow{\sim} \text{Ext}_{X \otimes_R^{\mathbb{L}} A}^i\left(\mathcal{E}, \varprojlim_r \mathcal{F}/m_A^r\right). \quad (3.19)$$

Again, we compute the Ext groups by using  $\mathcal{E}$  (which is degreewise projective) in the first variable and  $\mathcal{F}$  (which is degreewise flat) in the second variable. The obstruction for the functors  $\text{Ext}_{X \otimes_R^{\mathbb{L}} A}^i$  to commute with the inverse limit  $\varprojlim_r$  lies in the derived functor  $\varprojlim_r^1$ ;

however notice that the completeness assumption on  $A$  ensures that the tower  $A/m_A^r \rightarrow A/m_A^{r+1}$  satisfies the Mittag-Leffler condition (see [42] Section 3.5), and so does the induced tower  $\mathcal{F}/m_A^r \rightarrow \mathcal{F}/m_A^{r+1}$  (see [27] Section 4.2 for details). In particular we get

$$\varprojlim_r^1 \text{Ext}_{X \otimes_R^{\mathbb{L}} A}^{i-1}(\mathcal{F}, \mathcal{F}/m_A^r) = 0 \quad (3.20)$$

which implies

$$\text{Ext}_{X \otimes_R^{\mathbb{L}} A}^i(\mathcal{F}, \mathcal{F}) \xrightarrow{\sim} \varprojlim_r \text{Ext}_{X \otimes_R^{\mathbb{L}} A}^i(\mathcal{F}, \mathcal{F}/m_A^r) \quad \forall i \neq 1. \quad (3.21)$$

At last, we show the compatibility condition on components, i.e. we want to prove that the push-forward map

$$c(\pi_0 \mathbf{M}^n(A)) \longrightarrow \varprojlim_r c(\pi_0 \mathbf{M}^n(A/m_A^r)) \quad (3.22)$$

is bijective. This basically means to show that any inverse system

$$\left\{ \mathcal{E}_r \text{ s.t. } \mathcal{E}_r \text{ perfect complex of } \mathcal{O}_X \otimes_R^{\mathbb{L}} A/m_A^r\text{-modules} \right\}_{r \in \mathbb{N}}$$

determines uniquely a perfect  $\mathcal{O}_X \otimes_R^{\mathbb{L}} A$ -module in complexes via map (3.22); such a statement is precisely the version of Grothendieck Existence Theorem for perfect complexes: for a proof see [21] Theorem 3.2.2 or [9] Section 4.

It follows that Condition (7) holds as well, so this completes the proof.  $\square$

**Remark 3.25** Given a smooth and proper  $k$ -scheme  $X$ , consider the derived stack  $\mathbb{R}\text{Perf}_X$  defined by formula (3.3); clearly for all  $n \geq 0$  the derived geometric stack  $\mathbb{R}\text{Perf}_X^n$  is an open substack of  $\mathbb{R}\text{Perf}_X$  and moreover

$$\mathbb{R}\text{Perf}_X \simeq \bigcup_n \mathbb{R}\text{Perf}_X^n$$

so we recover the local geometricity of the stack  $\mathbb{R}Perf_X$  studied by Toën and Vaquié in [40].

### 3.3 Derived Moduli of Filtered Perfect Complexes

This section is devoted to the main result of this paper, that is the construction of a derived moduli stack  $\mathbb{R}Filt_X$  classifying filtered perfect complexes of  $\mathcal{O}_X$ -modules over some reasonable  $k$ -scheme  $X$ ; (local) geometricity of such a stack will be ensured by some quite natural cohomological finiteness conditions given in terms of the Rees construction (see Section 2.4): actually the very homotopy-theoretical features of the Rees functor collected in Theorem 2.30 will allow us to mimic most of the results and arguments of Section 3.2, which deal with the corresponding unfiltered situation.

In full analogy with what we did in Section 3.2, associate to any given derived scheme  $\mathcal{X}$  over  $R$  the cosimplicial differential graded commutative  $R$ -algebra  $\mathcal{O}(\mathcal{X})$  defined by formula (3.4).

**Definition 3.26** Define a *filtered derived module* over  $\mathcal{X}$  to be a cosimplicial filtered  $\mathcal{O}(\mathcal{X})$ -module in complexes.

More concretely Definition 3.26 says that a filtered derived module over  $\mathcal{X}$  is a pair  $(\mathcal{M}, F)$  made of filtered cochain complexes  $(\mathcal{M}^m, F)$  of  $\mathcal{O}(\mathcal{X})^m$ -modules related by maps

$$\begin{aligned} \partial^i : F^p \mathcal{M}^m \otimes_{\mathcal{O}(\mathcal{X})^m}^{\mathbb{L}} \mathcal{O}(\mathcal{X})^{m+1} &\longrightarrow F^p \mathcal{M}^{m+1} \\ \sigma^i : F^p \mathcal{M}^m \otimes_{\mathcal{O}(\mathcal{X})^m}^{\mathbb{L}} \mathcal{O}(\mathcal{X})^{m-1} &\longrightarrow F^p \mathcal{M}^{m-1} \end{aligned}$$

satisfying the usual cosimplicial identities and such that the diagrams

$$\begin{array}{ccc} \mathcal{M}_m \otimes_{\mathcal{O}(\mathcal{X})^m}^{\mathbb{L}} \mathcal{O}(\mathcal{X})^{m+1} & \xrightarrow{\quad} & \mathcal{M}_{m+1} \\ \uparrow & & \uparrow \\ F^1 \mathcal{M}_m \otimes_{\mathcal{O}(\mathcal{X})^m}^{\mathbb{L}} \mathcal{O}(\mathcal{X})^{m+1} & \xrightarrow{\quad} & F^1 \mathcal{M}_{m+1} \\ \uparrow & & \uparrow \\ F^2 \mathcal{M}_m \otimes_{\mathcal{O}(\mathcal{X})^m}^{\mathbb{L}} \mathcal{O}(\mathcal{X})^{m+1} & \xrightarrow{\quad} & F^2 \mathcal{M}_{m+1} \\ \uparrow & & \uparrow \\ \vdots & & \vdots \end{array} \quad \begin{array}{ccc} \mathcal{M}_{m-1} & \xleftarrow{\quad} & \mathcal{M}_m \otimes_{\mathcal{O}(\mathcal{X})^m}^{\mathbb{L}} \mathcal{O}(\mathcal{X})^{m-1} \\ \uparrow & & \uparrow \\ F^1 \mathcal{M}_{m-1} & \xleftarrow{\quad} & F^1 \mathcal{M}_m \otimes_{\mathcal{O}(\mathcal{X})^m}^{\mathbb{L}} \mathcal{O}(\mathcal{X})^{m-1} \\ \uparrow & & \uparrow \\ F^2 \mathcal{M}_{m-1} & \xleftarrow{\quad} & F^2 \mathcal{M}_m \otimes_{\mathcal{O}(\mathcal{X})^m}^{\mathbb{L}} \mathcal{O}(\mathcal{X})^{m-1} \\ \uparrow & & \uparrow \\ \vdots & & \vdots \end{array}$$

commute; in other words a derived filtered module  $(\mathcal{M}, F)$  is just a nested sequence

$$\cdots \hookrightarrow F^{p+1} \mathcal{M} \hookrightarrow F^p \mathcal{M} \hookrightarrow \cdots \hookrightarrow F^2 \mathcal{M} \hookrightarrow F^1 \mathcal{M} \hookrightarrow F^0 \mathcal{M} =: \mathcal{M}$$

in  $\mathfrak{dgMod}(\mathcal{X})$ . Notice that a filtered derived module is equipped with three different indexings, one coming from the filtration, one from the differential graded structure and the last one from the cosimplicial structure: a morphism of derived filtered modules will be an arrow preserving all of them, so there is a category of derived filtered modules on  $\mathcal{X}$ , which we will denote by  $\mathfrak{FdgMod}(\mathcal{X})$ .

Just like the unfiltered situation analysed in Section 3.2, observe that the projective model structure on filtered cochain complexes given by Theorem 2.18 induces a projective model

structure on  $\mathfrak{FdgMod}(\mathcal{X})$ ; in particular a morphism  $f : (\mathcal{M}, F) \rightarrow (\mathcal{N}, F)$  in  $\mathfrak{FdgMod}(\mathcal{X})$  is

- a weak equivalence if  $F^p f^m : F^p \mathcal{M}^m \rightarrow F^p \mathcal{N}^m$  is a quasi-isomorphism;
- a fibration if  $F^p f^m : F^p \mathcal{M}^m \rightarrow F^p \mathcal{N}^m$  is degreewise surjective;
- a cofibration if it has the left lifting property with respect to all fibrations.

There is also a natural simplicial structure on the category  $\mathfrak{FdgMod}(\mathcal{X})$  again coming from the simplicial structure on  $\mathfrak{FdgMod}_R$ : more clearly for any  $(\mathcal{M}, F), (\mathcal{N}, F) \in \mathfrak{FdgMod}(\mathcal{X})$  consider the chain complex  $(\text{Hom}_{\mathcal{X}}((\mathcal{M}, F), (\mathcal{N}, F)), \delta)$  defined by the relations

$$\begin{aligned} \text{Hom}_{\mathcal{X}}((\mathcal{M}, F), (\mathcal{N}, F))_n &:= \text{Hom}_{O(\mathcal{X})}((\mathcal{M}, F), (\mathcal{N}[-n], F)) \\ \forall (f, F) \in \text{Hom}_{\mathcal{X}}((\mathcal{M}, F), (\mathcal{N}, F))_n \quad \delta_n((f, F)) &\in \text{Hom}_{\mathcal{X}}((\mathcal{M}, F), (\mathcal{N}, F))_{n-1} \\ \text{defined by } F^p(\delta_n((f, F))) &:= F^p \bar{d}^n \circ F^p f - (-1)^n F^p f \circ F^p d^n \end{aligned} \quad (3.23)$$

and define the Hom spaces just by taking good truncation and denormalisation, i.e. set

$$\underline{\text{Hom}}_{\mathfrak{FdgMod}(\mathcal{X})}((\mathcal{M}, F), (\mathcal{N}, F)) := \mathbf{K}(\tau_{\geq 0} \text{Hom}_{\mathcal{X}}((\mathcal{M}, F), (\mathcal{N}, F))).$$

In a similar way, notice that the HOM complex for filtered derived modules defined by formula (3.23) sheafifies, so we have a well-defined  $\mathcal{H}om$ -sheaf bifunctor

$$\mathcal{H}om_{\mathcal{O}_{\mathcal{X}, \bullet}} : \mathfrak{FdgMod}^{op}(\mathcal{X}) \times \mathfrak{FdgMod}(\mathcal{X}) \longrightarrow \mathfrak{dgm}(\mathcal{X})$$

and consequently a derived  $\mathcal{H}om$  sheaf, given by the bifunctor

$$\begin{aligned} \mathbb{R}\mathcal{H}om_{\mathcal{O}_{\mathcal{X}, \bullet}} : \mathfrak{FdgMod}^{op}(\mathcal{X}) \times \mathfrak{FdgMod}(\mathcal{X}) &\longrightarrow \mathfrak{dgm}(\mathcal{X}) \\ ((\mathcal{M}, F), (\mathcal{N}, F)) &\longmapsto \mathcal{H}om(\mathbf{Q}((\mathcal{M}, F)), (\mathcal{N}, F)) \end{aligned}$$

where  $\mathbf{Q}((\mathcal{M}, F))$  is a functorial cofibrant replacement for  $(\mathcal{M}, F)$ . We can also define  $\mathcal{E}xt$  sheaves for the category  $\mathfrak{FdgMod}(\mathcal{X})$  by denoting for all  $(\mathcal{M}, F), (\mathcal{N}, F) \in \mathfrak{FdgMod}(\mathcal{O}_{\mathcal{X}, \bullet})$

$$\mathcal{E}xt_{\mathcal{O}_{\mathcal{X}, \bullet}}^i((\mathcal{M}, F), (\mathcal{N}, F)) := \mathcal{H}^i(\pi^0 \mathcal{X}, \mathbb{R}\mathcal{H}om_{\mathcal{O}_{\mathcal{X}, \bullet}}((\mathcal{M}, F), (\mathcal{N}, F))).$$

The Rees construction given by formula (2.15) readily extends to this context, as well; more formally consider the derived scheme  $\mathcal{X}[t]$  over  $R[t]$  whose Čech nerve is defined in simplicial degree  $m$  by the structure sheaf

$$\mathcal{O}_{\check{\mathcal{X}}[t]_m} := \mathcal{O}_{\check{\mathcal{X}}_m}[t]$$

so that its cosimplicial differential graded commutative  $R[t]$ -algebra of global sections is given in cosimplicial level  $m$  by

$$O(\mathcal{X}[t])^m := \Gamma(\check{\mathcal{X}}_m, \mathcal{O}_{\check{\mathcal{X}}_m})[t].$$

Again, there is a natural  $\mathbb{G}_m$ -action on the derived scheme  $\mathcal{X}[t]$  defined on rings of functions in level  $m$  as

$$\begin{aligned} \mathbb{G}_m \times O(\mathcal{X}[t])^m &\longrightarrow O(\mathcal{X}[t])^m \\ (\lambda, \varrho(t)) &\longmapsto \varrho(\lambda^{-1}t) \end{aligned} \quad (3.24)$$

therefore there is a category  $\mathbb{G}_m\text{-}\mathfrak{dgm}(\mathcal{X}[t])$  of graded derived modules over  $\mathcal{X}[t]$ , where the extra grading is induced by the action given by the formula (3.24). Clearly the  $\mathbb{G}_m$ -equivariant projective model structure determined by Theorem 2.27 extends to the category  $\mathbb{G}_m\text{-}\mathfrak{dgm}(\mathcal{X}[t])$ : in particular a morphism  $f : \mathcal{M} \rightarrow \mathcal{N}$  in  $\mathbb{G}_m\text{-}\mathfrak{dgm}(\mathcal{X}[t])$  is

- a weak equivalence if  $f^m : \mathcal{M}^m \rightarrow \mathcal{N}^m$  is a  $\mathbb{G}_m$ -equivariant quasi-isomorphism;
- a fibration if  $f^m : \mathcal{M}^m \rightarrow \mathcal{N}^m$  is a  $\mathbb{G}_m$ -equivariant degreewise surjection;
- a cofibration if it has the left lifting property with respect to all fibrations.

Now define the Rees module associated to the filtered derived module  $(\mathcal{M}, F)$  over  $\mathcal{X}$  to be the derived module  $\text{Rees}((\mathcal{M}, F))$  over  $\mathcal{X}[t]$  determined in cosimplicial level  $m$  by the (bigraded) complex of  $\mathcal{O}(\mathcal{X}[t])^m$ -modules

$$\text{Rees}((\mathcal{M}, F)) := \bigoplus_{p=0}^{\infty} F^p \mathcal{M}^m \cdot t^{-p}. \quad (3.25)$$

The construction (3.25) is clearly natural in all entries, so we get a functor

$$\text{Rees} : \mathfrak{FdgMod}(\mathcal{X}) \longrightarrow \mathbb{G}_m\text{-dgmMod}(\mathcal{X}[t]). \quad (3.26)$$

which is immediately seen to have – *mutatis mutandis* – all properties stated by Theorem 2.30. In particular, for all  $(\mathcal{M}, F), (N, F) \in \mathfrak{FdgMod}(\mathcal{X})$  define the groups

$$\text{Ext}_{\mathcal{X}}^{n-i}((\mathcal{M}, F), (N, F)) := \pi_i \underline{\text{Hom}}_{\mathfrak{FdgMod}(\mathcal{X})}((\mathcal{M}, F), (N[-n], F)) \quad (3.27)$$

and observe that Theorem 2.30.2 implies

$$\text{Ext}_{\mathcal{X}}^i((\mathcal{M}, F), (N, F)) = \text{Ext}_{\mathcal{X}[t]}^i(\text{Rees}((\mathcal{M}, F)), \text{Rees}((N, F)))^{\mathbb{G}_m}. \quad (3.28)$$

**Remark 3.27** Because of formula (3.28) and the exactness of the functor  $(-)^{\mathbb{G}_m}$ , we have that the local-to-global spectral sequence extends to the filtered context, i.e there is a convergent spectral sequence

$$H^p\left(\pi^0 \mathcal{X}, \mathcal{E}xt_{\mathcal{O}_{\mathcal{X},*}}^q((\mathcal{M}, F), (\mathcal{M}, F))\right) \Longrightarrow \text{Ext}_{\mathcal{X}}^{p+q}((\mathcal{M}, F), (\mathcal{M}, F)).$$

**Definition 3.28** Define a *filtered derived quasi-coherent sheaf* over  $\mathcal{X}$  to be a filtered derived module  $(\mathcal{M}, F)$  for which and  $F^p \mathcal{M} \in \text{dgmMod}_{\text{cart}}(\mathcal{X})$  for all  $p$ .

Denote by  $\mathfrak{FdgMod}_{\text{cart}}(\mathcal{X})$  the full subcategory of  $\mathfrak{FdgMod}(\mathcal{X})$  consisting of filtered quasi-coherent derived sheaves: the homotopy-theoretic properties of  $\mathfrak{FdgMod}(\mathcal{X})$  induce a simplicial structure and a well-behaved subcategory of weak equivalences on it.

**Remark 3.29** The Rees functor (3.26) respects quasi-coherence, meaning that it restricts to a functor

$$\text{Rees} : \mathfrak{FdgMod}_{\text{cart}}(\mathcal{X}) \longrightarrow \mathbb{G}_m\text{-dgmMod}_{\text{cart}}(\mathcal{X}[t]).$$

which obviously still maps weak equivalences to weak equivalences.

Now our goal is to study derived moduli of filtered derived quasi-coherent sheaves by means of Lurie-Pridham representability: in order to reach this we will literally follow the strategy described in Section 3.2 where moduli of unfiltered complexes were tackled; in particular we will prove filtered analogues of Proposition 3.22, Corollary 3.23 and Theorem 3.24. In the following, given any filtered derived quasi-coherent sheaf  $(\mathcal{E}, F)$  over some derived geometric stack denote by  $\hat{F}$  the filtration induced by (derived) base-change and by  $\tilde{F}$  the one induced on quotients.

From now on fix  $R$  to be an ordinary (underived)  $k$ -algebra and  $X$  to be a quasi-compact semi-separated scheme over  $R$ ; define the functor

$$\begin{aligned} Fd\mathcal{CART}_X : \mathfrak{dg}_b\mathfrak{Nil}_R^{\leq 0} &\longrightarrow \mathfrak{Set} \\ A &\longmapsto \left( \mathfrak{FdgMod}_{\mathcal{CART}} \left( X \otimes_R^{\mathbb{L}} A \right) \right)^c \end{aligned} \quad (3.29)$$

where again  $\left( \mathfrak{FdgMod}_{\mathcal{CART}} \left( X \otimes_R^{\mathbb{L}} A \right) \right)^c$  is the full simplicial subcategory of  $\mathfrak{FdgMod}_{\mathcal{CART}} \left( X \otimes_R^{\mathbb{L}} A \right)$  on cofibrant objects.

**Lemma 3.30** *Let  $f : A \rightarrow B$  a square-zero extension in  $\mathfrak{dgAlg}_R^{\leq 0}$ ; then the induced morphism*

$$\begin{aligned} \mathbf{f} : A[t] &\longrightarrow B[t] \\ A_n[t] \ni \sum_i a_i t^i &\xrightarrow{f_n} \sum_i f(a_i) t^i \in B_n[t] \end{aligned}$$

*is a square-zero extension in  $\mathfrak{dgAlg}_R^{\leq 0}$ . Moreover  $\mathbf{f}$  is acyclic whenever so is  $f$ .*

*Proof* Denote  $I := \ker(f)$ ; then  $\ker(\mathbf{f}) = I[t]$ , where

$$I[t] : \quad \cdots \xrightarrow{d} I_2[t] \xrightarrow{d} I_1[t] \xrightarrow{d} I_0[t].$$

In particular  $I[t]^2 = 0 \Leftrightarrow I^2 = 0$  and  $H^i(I[t]) = 0 \Leftrightarrow H^i(I) = 0$ .  $\square$

**Proposition 3.31** *Functor (3.29) is 2-homogeneous and formally 2-quasi-smooth.*

*Proof* The argument of Proposition 3.22 applies to this context as well, we sketch the main adjustments.

In order to verify that  $Fd\mathcal{CART}_X$  is 2-homogeneous take a square-zero extension  $A \rightarrow B$  and a morphism  $C \rightarrow B$  in  $\mathfrak{dg}_b\mathfrak{Nil}_R^{\leq 0}$  and fix  $(\mathcal{E}, F), (\mathcal{E}', F) \in Fd\mathcal{CART}_X(A \times_B C)$ . Cofibrancy of such pairs – which by Proposition 2.21 implies filtration-levelwise degree-wise projectiveness – ensures that the commutative square of simplicial sets

$$\begin{array}{ccc} \underline{\mathrm{Hom}}_{Fd\mathcal{CART}(A \times_B C)}((\mathcal{E}, F), (\mathcal{E}', F)) & \longrightarrow & \underline{\mathrm{Hom}}_{Fd\mathcal{CART}(A)}((\mathcal{E} \otimes_{A \times_B C} A, \hat{F}), (\mathcal{E}' \otimes_{A \times_B C} A, \hat{F})) \\ \downarrow & & \downarrow \\ \underline{\mathrm{Hom}}_{Fd\mathcal{CART}(C)}((\mathcal{E} \otimes_{A \times_B C} C, \hat{F}), (\mathcal{E}' \otimes_{A \times_B C} C, \hat{F})) & \longrightarrow & \underline{\mathrm{Hom}}_{Fd\mathcal{CART}(B)}((\mathcal{E} \otimes_{A \times_B C} B, \hat{F}), (\mathcal{E}' \otimes_{A \times_B C} B, \hat{F})) \end{array}$$

is actually Cartesian. Then fix  $(\mathcal{E}_A, F_A) \in Fd\mathcal{CART}_X(A)$  and  $(\mathcal{E}_C, F_C) \in Fd\mathcal{CART}_X(C)$ , let  $\alpha : (\mathcal{E}_A \otimes_A B, \hat{F}_A) \rightarrow (\mathcal{E}_C \otimes_C B, \hat{F}_C)$  be a filtered isomorphism and define

$$(\mathcal{E}, F) := (\mathcal{E}_A \otimes_{\alpha, \mathcal{E}_C \otimes_C B} \mathcal{E}_C, F_A \otimes F_C) \simeq (\mathcal{E}_C \otimes_{\alpha, \mathcal{E}_A \otimes B} \mathcal{E}_A, F_C \otimes F_A). \quad (3.30)$$

The filtered derived module  $(\mathcal{E}, F)$  is actually a cofibrant filtered derived quasi-coherent sheaf on  $X \otimes_R (A \otimes_B C)$ , namely  $(\mathcal{E}, F) \in Fd\mathcal{CART}_X(A \times_B C)$ ; we also have that

$$(\mathcal{E}, F) \otimes_{A \times_B C} A \simeq (\mathcal{E}_A, F_A) \quad (\mathcal{E}, F) \otimes_{A \times_B C} C \simeq (\mathcal{E}_C, F_C)$$

and this completes the proof that  $Fd\mathcal{CART}_X$  is a 2-homogeneous functor.

Now we want to prove that the functor  $Fd\mathcal{CART}_X$  is formally 2-quasi-smooth, so we start by showing that  $\underline{\mathrm{Hom}}_{Fd\mathcal{CART}_X}$  is formally quasi-smooth; for this reason take a square-zero extension  $A \rightarrow B$  in  $\mathfrak{d}\mathfrak{g}_b\mathfrak{N}\mathfrak{il}_R^{\leq 0}$  and consider the induced  $R[t]$ -linear morphism  $A \rightarrow B$ , as done in Lemma 3.30. Let  $(\mathcal{E}, F), (\mathcal{E}', F) \in Fd\mathcal{CART}(A)$  and look at the induced morphism of simplicial sets

$$\underline{\mathrm{Hom}}_{Fd\mathcal{CART}(A)}((\mathcal{E}, F), (\mathcal{E}', F)) \longrightarrow \underline{\mathrm{Hom}}_{Fd\mathcal{CART}(B)}((\mathcal{E} \otimes_A B, \hat{F}), (\mathcal{E}' \otimes_A B, \hat{F})). \quad (3.31)$$

By Theorem 2.30.2, map (3.31) is a (trivial) fibration if and only if the map

$$\begin{array}{c} \underline{\mathrm{Hom}}_{d\mathcal{CART}(A[t])}(\mathrm{Rees}((\mathcal{E}, F)), \mathrm{Rees}((\mathcal{E}', F)))^{\mathbb{G}_m} \\ \downarrow \\ \underline{\mathrm{Hom}}_{d\mathcal{CART}(B[t])}(\mathrm{Rees}((\mathcal{E} \otimes_A B, \hat{F})), \mathrm{Rees}((\mathcal{E}' \otimes_A B, \hat{F})))^{\mathbb{G}_m} \end{array}$$

is a (trivial) fibration, which in turn is equivalent to say that

$$\begin{array}{c} \underline{\mathrm{Hom}}_{d\mathcal{CART}(A[t])}(\mathrm{Rees}((\mathcal{E}, F)), \mathrm{Rees}((\mathcal{E}', F))) \\ \downarrow \\ \underline{\mathrm{Hom}}_{d\mathcal{CART}(B[t])}(\mathrm{Rees}((\mathcal{E} \otimes_A B, \hat{F})), \mathrm{Rees}((\mathcal{E}' \otimes_A B, \hat{F}))) \end{array} \quad (3.32)$$

is a (trivial) fibration, as functor  $(-)^{\mathbb{G}_m}$  is exact. Now by Lemma 3.30 we have that the morphism of  $R[t]$ -algebras  $A[t] \rightarrow B[t]$  is a square-zero extension that is acyclic whenever so is  $A \rightarrow B$ , while Proposition 3.22 ensures that map (3.32) is a fibration which is trivial if  $A[t] \rightarrow B[t]$  is acyclic: these observations conclude the proof of the formal quasi-smoothness of  $\underline{\mathrm{Hom}}_{Fd\mathcal{CART}_X}$ . In order to finish the proof, it only remains to show that the base-change morphism

$$Fd\mathcal{CART}_X(A) \longrightarrow Fd\mathcal{CART}_X(B) \quad (3.33)$$

is a 2-fibration, which should be trivial whenever the square-zero extension  $A \rightarrow B$  is acyclic. Note first that the computations in [25] Section 7, together with the definition of Ext groups for filtered derived modules given by formula (3.27) and the isomorphism provided by formula (3.28), imply that obstructions to lifting a filtered quasi-coherent module  $(\mathcal{E}, F) \in Fd\mathcal{CART}_X(B)$  to  $Fd\mathcal{CART}_X(A)$  lie in the group

$$\mathrm{Ext}_{X \otimes_R^{\mathbb{L}} B}^2((\mathcal{E}, F), (\mathcal{E} \otimes_B I, \hat{F})) \simeq \mathrm{Ext}_{(X \otimes_R^{\mathbb{L}} B)[t]}^2(\mathrm{Rees}((\mathcal{E}, F)), \mathrm{Rees}((\mathcal{E} \otimes_B I, \hat{F})))^{\mathbb{G}_m}.$$

so if  $H^*(I) = 0$  then map (3.33) is a trivial 2-fibration. Now fix  $(\mathcal{E}, F) \in Fd\mathcal{CART}_X(A)$ ,  $(\mathcal{H}, G) \in Fd\mathcal{CART}_X(B)$  and let  $\theta : (\mathcal{E} \otimes_A B, \hat{F}) \rightarrow (\mathcal{H}, G)$  be a homotopy equivalence in  $Fd\mathcal{CART}_X(B)$ : we want to prove that  $\theta$  lifts to a homotopy equivalence  $\check{\theta} : (\mathcal{E}, F) \rightarrow (\check{\mathcal{H}}, \check{G})$  in  $Fd\mathcal{CART}_X(A)$ . Apply the Rees functor (3.26) to all data: by Theorem 2.30 we end up to be given a homotopy equivalence

$$\mathrm{Rees}(\theta) : \mathrm{Rees}((\mathcal{E} \otimes_A B, \hat{F})) \longrightarrow \mathrm{Rees}((\mathcal{H}, G))$$

in  $d\mathcal{CART}_X(B[t])$  which by Proposition 3.22 lifts to a homotopy equivalence in  $d\mathcal{CART}_X(A[t])$ ; in particular this ensures that suitable lifts  $\check{\theta}$  of the homotopy equivalence  $\theta$  do exist, again by Theorem 2.30. This completes the proof.  $\square$

We can build upon Proposition 3.31 a filtered version of Corollary 3.23.

**Corollary 3.32** *Let*

$$\mathbf{M} : \mathfrak{Alg}_R \rightarrow \mathfrak{sCat}$$

*be a presheaf satisfying the following conditions:*

1.  $\mathbf{M}$  is a  $n$ -truncated hypersheaf;
2.  $\mathbf{M}$  is open in the functor

$$A \mapsto \pi^0 \mathcal{W} \left( \mathfrak{FdgMod} \left( X \otimes_R^{\mathbb{L}} A \right)_{\text{cart}} \right) \quad A \in \mathfrak{Alg}_R;$$

3. If  $\{f_\alpha : A \rightarrow B_\alpha\}_\alpha$  is an étale cover, then  $(\mathcal{E}, F) \in \pi_0 \pi^0 \mathcal{W} \left( \mathfrak{FdgMod} \left( X \otimes_R^{\mathbb{L}} A \right)_{\text{cart}} \right)$  lies in the essential image of  $\pi_0 \mathbf{M}(A)$  whenever  $\left( (f_\alpha)^* \mathcal{E}, \hat{F} \right)$  is in the essential image of  $\pi_0 \mathbf{M}(B_\alpha)$  for all  $\alpha$ ;
4. For all finitely generated  $A \in \mathfrak{Alg}_R$  and all  $(\mathcal{E}, F) \in \mathbf{M}(A)$ , the functors

$$\text{Ext}_{X \otimes_R^{\mathbb{L}} A}^i \left( (\mathcal{E}, F), \left( \mathcal{E} \otimes_A^{\mathbb{L}} -, \hat{F} \right) \right) : \mathfrak{Mod}_A \longrightarrow \mathfrak{Ab}$$

*preserve filtered colimits*  $\forall i \neq 1$ ;

5. For all finitely generated integral domains  $A \in \mathfrak{Alg}_R$  and all  $(\mathcal{E}, F) \in \mathbf{M}(A)$ , the groups  $\text{Ext}_{X \otimes_R^{\mathbb{L}} A}^i \left( (\mathcal{E}, F), (\mathcal{E}, F) \right)$  are finitely generated  $A$ -modules;
6. The functor

$$c(\pi_0 \mathbf{M}) : \mathfrak{Alg}_R \longrightarrow \mathfrak{Set}$$

*of components of the groupoid*  $\pi_0 \mathbf{M}$  *preserves filtered colimits*;

7. For all complete discrete local Noetherian normal  $R$ -algebras  $A$ , all  $(\mathcal{E}, F) \in \mathbf{M}(A)$  and all  $i > 0$  the canonical maps

$$c(\pi_0 \mathbf{M}(A)) \longrightarrow \varprojlim_r c(\pi_0 \mathbf{M}(A/\mathfrak{m}_A^r))$$

$$\text{Ext}_{X \otimes_R^{\mathbb{L}} A}^i \left( (\mathcal{E}, F), (\mathcal{E}, F) \right) \longrightarrow \varprojlim_r \text{Ext}_{\mathcal{F} \otimes_R^{\mathbb{L}} A}^i \left( (\mathcal{E}, F), \left( \mathcal{E}/\mathfrak{m}_A^r, \tilde{F} \right) \right) \quad \forall i < 0$$

*are isomorphisms.*

*Let*

$$\check{\mathbf{M}} : \mathfrak{dg}_b \mathfrak{Nil}_R^{\leq 0} \longrightarrow \mathfrak{sCat}$$

*be the full simplicial subcategory of*  $\mathcal{W}(\text{FdCART}_X(A))$  *consisting of objects*  $(\mathcal{F}, F)$  *for which the pair*  $\left( \mathcal{F} \otimes_A H^0(A), \hat{F} \right)$  *is weakly equivalent in*  $\mathfrak{FdgMod}_{\text{cart}} \left( X \otimes_R^{\mathbb{L}} H^0(A) \right)$  *to an object of*  $\mathbf{M}(H^0(A))$ . *Then the functor*  $\bar{W}\check{\mathbf{M}}$  *is (the restriction to*  $\mathfrak{dg}_b \mathfrak{Nil}_R^{\leq 0}$  *of) a derived geometric*  $n$ -*stack.*

*Proof* The same arguments used to prove Corollary 3.23 carry over to this context, using Proposition 3.31 in place of Proposition 3.22 and observing – as done in the proof of Proposition 3.31 itself – that

$$\begin{aligned} D_{(\mathcal{E}, F)}^i \left( \bar{W}\check{\mathbf{M}}, M \right) &\simeq \text{Ext}_{X \otimes_A^{\mathbb{L}} M}^{i+1} \left( (\mathcal{E}, F), \left( \mathcal{E} \otimes_A^{\mathbb{L}} M, \hat{F} \right) \right) \\ &\simeq \text{Ext}_{(X \otimes_A^{\mathbb{L}} M)[t]} \left( \text{Rees}((\mathcal{E}, F)), \text{Rees} \left( \left( \mathcal{E} \otimes_A^{\mathbb{L}} M, \hat{F} \right) \right) \right)^{\mathbb{G}_m}. \end{aligned}$$



Also Condition (2) tells us that

$$\check{\mathbf{M}}(A) \approx \mathbf{M}\left(H^0(A)\right) \times_{\mathcal{W}(\mathfrak{F}\mathfrak{d}\mathfrak{g}\mathfrak{M}\mathfrak{o}\mathfrak{d}(X \otimes_R^{\mathbb{L}} H^0(A))_{\text{cart}})}^h \mathcal{W}\left(\mathfrak{F}\mathfrak{d}\mathfrak{g}\mathfrak{M}\mathfrak{o}\mathfrak{d}\left(X \otimes_R^{\mathbb{L}} A\right)_{\text{cart}}\right)$$

which is the filtered analogue of formula (3.8).

The only claim which still needs to be verified is the one saying that  $\bar{W}\mathcal{W}(\mathbf{M})$  is an étale hypersheaf: observe that, by combining Condition (3) and Proposition 3.16, this amounts to check that  $\bar{W}\mathcal{W}(Fd\text{CART}_X)$  is a hypersheaf for the étale topology, thus fix an étale hypercover  $B \rightarrow B^\bullet$  and consider the induced map

$$\bar{W}\mathcal{W}(Fd\text{CART}_X)(B) \longrightarrow \varprojlim_n \left(\bar{W}\mathcal{W}(Fd\text{CART}_X)(B^\bullet)\right). \quad (3.34)$$

Let us apply the Rees construction to map (3.34): by Remark 3.29 the Rees functor (3.26) descends to quasi-coherent modules and as a consequence of Theorem 2.30 it preserves cofibrant objects, so map (3.34) becomes

$$\bar{W}\mathcal{W}(d\text{CART}_{X[t]})(B[t]) \longrightarrow \varprojlim_n \left(\bar{W}\mathcal{W}(d\text{CART}_{X[t]})(B[t]^\bullet)\right) \quad (3.35)$$

and map (3.35) is actually a weak equivalence because  $\bar{W}\mathcal{W}(d\text{CART}_{X[t]})$  is a hypersheaf for the étale topology over  $\mathfrak{d}\mathfrak{g}\mathfrak{A}\mathfrak{I}\mathfrak{g}_{R[t]}^{\leq 0}$ , as observed in the proof of Corollary 3.23. Arguing backwards, this implies that  $\bar{W}\mathcal{W}(Fd\text{CART}_X)$  is itself an étale hypersheaf, and this completes the proof.  $\square$

Now we are ready to discuss derived moduli of filtered perfect complexes; for this reason consider the functor

$$\begin{aligned} \mathbf{M}_{\text{filt}}^n : \mathfrak{A}\mathfrak{I}\mathfrak{g}_R &\longrightarrow \mathfrak{s}\mathfrak{C}\mathfrak{a}\mathfrak{t} \\ A &\longmapsto \mathbf{M}_{\text{filt}}^n(A) := \text{full simplicial subcategory of filtered} \\ &\quad \text{complexes } (\mathcal{E}, F) \text{ of } \mathcal{O}_X \otimes_R^{\mathbb{L}} A\text{-modules such that:} \\ &\quad a) F \text{ is bounded below} \\ &\quad b) F^p \mathcal{E} \text{ is perfect for all } p \\ &\quad c) \text{Ext}_{X \otimes_R^{\mathbb{L}} A}^i((\mathcal{E}, F), (\mathcal{E}, F)) = 0 \text{ for } i < -n \end{aligned} \quad (3.36)$$

which classifies filtered perfect  $\mathcal{O}_X$ -modules in complexes with trivial Ext groups in higher negative degrees.

**Theorem 3.33** *In the above notations, assume that the scheme  $X$  is smooth and proper; then functor (3.36) induces a derived geometric  $n$ -stack  $\mathbb{R}\text{Filt}_X^n$ .*

*Proof* We have to prove that functor (3.36) satisfies the conditions of Corollary 3.32: again our strategy consists of adapting the proof of Theorem 3.24 to the filtered case by means of the homotopy-theoretical properties of the Rees construction.

First of all, notice the vanishing assumption about the Ext groups given by Axiom (c) corresponds exactly to the  $n$ -truncation of the presheaf  $\mathbf{M}_{\text{filt}}^n$ , which gives us Condition (1).

As regards Condition (2), let us show the openness of  $\mathbf{M}_{\text{filt}}^n$  inside  $\pi^0 \mathcal{W}(\mathfrak{F}\mathfrak{d}\mathfrak{g}\mathfrak{M}\mathfrak{o}\mathfrak{d}(X \otimes_R^{\mathbb{L}} -)_{\text{cart}})$ , which essentially amounts to prove that the morphism of formal groupoids

$$\pi_0 \mathbf{M}_{\text{filt}}^n \hookrightarrow \pi_0 \pi^0 \mathcal{W}\left(\mathfrak{d}\mathfrak{g}\mathfrak{M}\mathfrak{o}\mathfrak{d}\left(X \otimes_R^{\mathbb{L}} -\right)_{\text{cart}}\right) \quad (3.37)$$

is formally étale. Fix a square-zero extension  $I \hookrightarrow A \twoheadrightarrow B$  and an object  $(\mathcal{E}, F) \in \mathbf{M}_{\text{filt}}^n(B)$ , then look at the maps which morphism (3.37) induces on tangent and obstruction spaces; by combining the results in [18] Section 4 and [27] Theorem 4.12 about the Deformation Theory of perfect complexes and quasi-coherent modules respectively with the homotopy-theoretical features of the Rees functor established by Theorem 2.30 and formula (3.28) we have that

- the tangent space to the functor  $\pi_0 \mathbf{M}_{\text{filt}}^n$  at  $(\mathcal{E}, F)$  is given by

$$\begin{aligned} & \text{Ext}_{X \otimes_R^{\mathbb{L}} A}^1 \left( (\mathcal{E}, F), \left( \mathcal{E} \otimes_B^{\mathbb{L}} I, \hat{F} \right) \right) \\ & \simeq \text{Ext}_{(X \otimes_R^{\mathbb{L}} A)[t]}^1 \left( \text{Rees}((\mathcal{E}, F)), \text{Rees} \left( \left( \mathcal{E} \otimes_B^{\mathbb{L}} I, \hat{F} \right) \right) \right)^{\mathbb{G}_m} \end{aligned}$$

- a functorial obstruction space for  $\pi_0 \mathbf{M}_{\text{filt}}^n$  at  $(\mathcal{E}, F)$  is given by

$$\begin{aligned} & \text{Ext}_{X \otimes_R^{\mathbb{L}} A}^2 \left( (\mathcal{E}, F), \left( \mathcal{E} \otimes_B^{\mathbb{L}} I, \hat{F} \right) \right) \\ & \simeq \text{Ext}_{(X \otimes_R^{\mathbb{L}} A)[t]}^2 \left( \text{Rees}((\mathcal{E}, F)), \text{Rees} \left( \left( \mathcal{E} \otimes_B^{\mathbb{L}} I, \hat{F} \right) \right) \right)^{\mathbb{G}_m} \end{aligned}$$

- the tangent space to the functor  $\pi_0 \pi^0 \mathcal{W}(\mathfrak{FdgMod}(X \otimes_R^{\mathbb{L}} -)_{\text{cart}})$  at  $(\mathcal{E}, F)$  is given by the group

$$\begin{aligned} & \text{Ext}_{X \otimes_R^{\mathbb{L}} A}^1 \left( (\mathcal{E}, F), \left( \mathcal{E} \otimes_B^{\mathbb{L}} I, \hat{F} \right) \right) \\ & \simeq \text{Ext}_{(X \otimes_R^{\mathbb{L}} A)[t]}^1 \left( \text{Rees}((\mathcal{E}, F)), \text{Rees} \left( \left( \mathcal{E} \otimes_B^{\mathbb{L}} I, \hat{F} \right) \right) \right)^{\mathbb{G}_m} \end{aligned}$$

- a functorial obstruction space for  $\pi_0 \pi^0 \mathcal{W}(\mathfrak{FdgMod}(X \otimes_R^{\mathbb{L}} -)_{\text{cart}})$  at  $(\mathcal{E}, F)$  is given by

$$\begin{aligned} & \text{Ext}_{X \otimes_R^{\mathbb{L}} A}^2 \left( (\mathcal{E}, F), \left( \mathcal{E} \otimes_B^{\mathbb{L}} I, \hat{F} \right) \right) \\ & \simeq \text{Ext}_{(X \otimes_R^{\mathbb{L}} A)[t]}^2 \left( \text{Rees}((\mathcal{E}, F)), \text{Rees} \left( \left( \mathcal{E} \otimes_B^{\mathbb{L}} I, \hat{F} \right) \right) \right)^{\mathbb{G}_m} \end{aligned}$$

so the group homomorphisms induced on first-order deformations and obstruction theories is just identities, therefore Condition (2) holds.

In terms of Condition (3), notice that the argument showing the analogous claim in the proof of Theorem 3.24 also holds in this context, since the filtered complexes we are parametrisng are perfect in each level of the filtration; thus Condition (3) holds.

In order to check Condition (4), fix a finitely generated  $R$ -algebra  $A$  and a pair  $(\mathcal{E}, F) \in \mathbf{M}_{\text{filt}}^n(A)$  and consider an inductive system  $\{B_\alpha\}_\alpha$  of  $A$ -algebras. Since  $F^m \mathcal{E}$  is perfect for

any  $m$ , we can choose a “flat” resolution (see Theorem 3.24 for more explanation)  $(\mathcal{F}, \hat{F})$  for the filtered complex  $(\mathcal{E}, F)$ ; therefore there is a chain of isomorphisms

$$\begin{aligned}
 & \operatorname{Ext}_{X \otimes_R A}^i \left( (\mathcal{E}, F), \left( \mathcal{E} \otimes_A^{\mathbb{L}} \varinjlim_{\alpha} B_{\alpha}, \hat{F} \right) \right) \\
 & \operatorname{Ext}_{X \otimes_R A}^i \left( (\mathcal{E}, F), \left( \mathcal{F} \otimes_A \varinjlim_{\alpha} B_{\alpha}, \hat{F} \right) \right) \\
 & \simeq \operatorname{Ext}_{(X \otimes_R A)[t]}^i \left( \operatorname{Rees}(\mathcal{E}, F), \operatorname{Rees} \left( \mathcal{F} \otimes_A \varinjlim_{\alpha} B_{\alpha}, \hat{F} \right) \right)^{\mathbb{G}_m} \\
 & \simeq \operatorname{Ext}_{(X \otimes_R A)[t]}^i \left( \operatorname{Rees}(\mathcal{E}, F), \operatorname{Rees} \left( \varinjlim_{\alpha} (\mathcal{F} \otimes_A B_{\alpha}, \hat{F}_{\alpha}) \right) \right)^{\mathbb{G}_m} \\
 & \simeq \operatorname{Ext}_{(X \otimes_R A)[t]}^i \left( \operatorname{Rees}(\mathcal{E}, F), \varinjlim_{\alpha} \operatorname{Rees}(\mathcal{F} \otimes_A B_{\alpha}, \hat{F}_{\alpha}) \right)^{\mathbb{G}_m} \\
 & \simeq \left( \varinjlim_{\alpha} \operatorname{Ext}_{(X \otimes_R A)[t]}^i \left( \operatorname{Rees}(\mathcal{E}, F), \operatorname{Rees}(\mathcal{F} \otimes_A B_{\alpha}, \hat{F}_{\alpha}) \right) \right)^{\mathbb{G}_m} \\
 & \simeq \varinjlim_{\alpha} \operatorname{Ext}_{(X \otimes_R A)[t]}^i \left( \operatorname{Rees}(\mathcal{E}, F), \operatorname{Rees}(\mathcal{F} \otimes_A B_{\alpha}, \hat{F}_{\alpha}) \right)^{\mathbb{G}_m} \\
 & \simeq \varinjlim_{\alpha} \operatorname{Ext}_{X \otimes_R A}^i \left( (\mathcal{E}, F), (\mathcal{F} \otimes_A B_{\alpha}, \hat{F}_{\alpha}) \right) \\
 & \simeq \varinjlim_{\alpha} \operatorname{Ext}_{X \otimes_R A}^i \left( (\mathcal{E}, F), (\mathcal{E} \otimes_A^{\mathbb{L}} B_{\alpha}, \hat{F}_{\alpha}) \right)
 \end{aligned}$$

where we have used the various properties collected in Theorem 2.30, the induced description of the Ext groups determined by formula (3.28), the exactness of the functor  $(-)^{\mathbb{G}_m}$  and the filtration-levelwise degreewise flatness of the representative  $(\mathcal{F}, \hat{F})$ . It follows that Condition (4) holds.

The way we prove Condition (5) is exactly the same utilised to show the corresponding claim in Theorem 3.24: indeed, note that such an argument carries over to this context, provided that we use the “filtered version” of the local-to-global spectral sequence given by Remark 3.27 in place of the classical one; thus Condition (5) holds.

Now we look at Condition (6); fix an inductive system  $\{A_{\alpha}\}_{\alpha}$  of  $R$ -algebras and let  $A := \varinjlim_{\alpha} A_{\alpha}$ : we need to show that

$$c(\pi_0 \mathbf{M}_{\text{filt}}^n(A)) = \varinjlim_{\alpha} c(\pi_0 \mathbf{M}_{\text{filt}}^n(A_{\alpha})) \quad (3.38)$$

where for any  $R$ -algebra  $B$

$$\begin{aligned}
 & c(\pi_0 \mathbf{M}_{\text{filt}}^n(B)) \\
 & := \left\{ \text{isomorphism classes of filtered perfect complexes of } (\mathcal{O}_X \otimes_R^{\mathbb{L}} B)\text{-modules} \right\}.
 \end{aligned} \quad (3.39)$$

According to formula (3.39) an element in  $\lim_{\alpha} c(\pi_0 \mathbf{M}_{\text{filt}}^n(A_\alpha))$  consists of a direct system of classes

$$\left\{ \left[ \begin{array}{c} \mathcal{E}_\alpha \\ \uparrow \\ F^1 \mathcal{E}_\alpha \\ \uparrow \\ F^2 \mathcal{E}_\alpha \\ \uparrow \\ \vdots \end{array} \right] \right\}_\alpha \quad (3.40)$$

where for all  $p$  and all  $\alpha$   $F^p \mathcal{E}_\alpha$  is a perfect complex of  $\mathcal{O}_X \otimes_R^{\mathbb{L}} A_\alpha$ -modules. In the proof of Theorem 3.24 we have shown that each system  $\{[F^p \mathcal{E}_\alpha]\}_\alpha$  determines uniquely an isomorphism class of perfect  $\mathcal{O}_X \otimes_R^{\mathbb{L}} A$ -module in complexes and notice that inclusions are preserved under inductive limits, thus the object described by formula (3.43) determines a unique class in  $c(\pi_0 \mathbf{M}_{\text{filt}}^n(A))$ , which means that formula (3.39) is verified. It follows that Condition (6) holds.

Lastly, we have to check Condition (7), so fix a complete discrete local Noetherian  $R$ -algebra  $A$  and a pair  $(\mathcal{E}, F) \in \mathbf{M}_{\text{filt}}^n(A)$ .

Consider for all  $i < 0$  the canonical map

$$\text{Ext}_{X \otimes_R^{\mathbb{L}} A}^i((\mathcal{E}, F), (\mathcal{E}, F)) \longrightarrow \lim_{\leftarrow r} \text{Ext}_{\mathcal{F} \otimes_R^{\mathbb{L}} A}^i((\mathcal{E}, F), (\mathcal{E}/\mathfrak{m}_A^r, \tilde{F})) \quad (3.41)$$

induced by the isomorphism

$$\hat{A} := \lim_{\leftarrow r} A/\mathfrak{m}_A^r \quad (3.42)$$

to the pronilpotent completion of  $A$ . Now by formula (3.28) we see that map (3.41) is the same as the group morphism

$$\begin{array}{c} \text{Ext}_{(X \otimes_R^{\mathbb{L}} A)[t]}^i(\text{Rees}((\mathcal{E}, F)), \text{Rees}((\mathcal{E}, F)))^{\mathbb{G}_m} \\ \downarrow \\ \lim_{\leftarrow r} \text{Ext}_{(\mathcal{F} \otimes_R^{\mathbb{L}} A)[t]}^i(\text{Rees}((\mathcal{E}, F)), \text{Rees}((\mathcal{E}/\mathfrak{m}_A^r, \tilde{F})))^{\mathbb{G}_m} \end{array}$$

which is an isomorphism, as follows by combining the exactness of the functor  $(-)^{\mathbb{G}_m}$  and the computations in the proof of Theorem 3.24.

At last, the compatibility condition on the components is easily checked by using techniques similar to the ones utilised to verify Condition (6). As a matter of fact take any inverse system

$$\left\{ \begin{array}{c} \mathcal{E}_r \\ \uparrow \\ F^1 \mathcal{E}_r \\ \uparrow \\ F^2 \mathcal{E}_r \\ \uparrow \\ \vdots \end{array} \right\}_r \quad (3.43)$$

of filtered perfect complexes of  $\mathcal{O}_X \otimes_R^{\mathbb{L}} A/\mathfrak{m}_A^r$ -modules and note that the proof of the corresponding statement in Theorem 3.24 allows us to lift each level  $F^p \mathcal{E}_r$  to a perfect complex of  $\mathcal{O}_X \otimes_R^{\mathbb{L}} A$ -modules; moreover countable limits preserve inclusions: this concludes the verification of the claim.

It follows that Condition (7) holds, so the proof is complete.  $\square$

Now define

$$\mathbb{R}Filt_X := \bigcup_n \mathbb{R}Filt_X^n$$

which is the simplicial presheaf parametrising filtered perfect complexes over the scheme  $X$ : Theorem 3.33 ensures that – if  $X$  is smooth and proper –  $\mathbb{R}Filt_X$  is a locally geometric derived stack over  $R$ ; this comment provides the ultimate comparison between moduli of complexes and moduli of filtered complexes.

*Remark 3.34* An interesting derived substack of  $\mathbb{R}Filt_X$  is the *stack of submodules* over  $X$ , which we denote by  $\mathbb{R}Sub_X$ ; this is the simplicial presheaf over  $\mathfrak{dgAlg}_k^{\leq 0}$  parametrising filtered perfect  $\mathcal{O}_X$ -modules in complexes  $(\mathcal{E}, F)$  such that the filtration  $F$  has length 2: in other words the functor  $\mathbb{R}Sub_X$  classifies pairs made of a perfect complex  $\mathcal{E}$  and a subcomplex  $F^1 \mathcal{E}$ , which is perfect as well.  $\mathbb{R}Sub_X$  is clearly a derived substack of  $\mathbb{R}Filt_X$  and it is also locally geometric when  $X$  is smooth and proper; as a matter of fact consider the simplicial presheaf  $\mathbb{R}Sub_X^n$  parametrising filtered  $\mathcal{O}_X$ -modules in complexes for which the filtration has length 2 and the relevant higher Ext groups vanish: then the arguments and techniques explained in this section show that  $\mathbb{R}Sub_X^n$  is a derived geometric  $n$ -stack over  $R$  and moreover we have that

$$\mathbb{R}Sub_X = \bigcup_n \mathbb{R}Sub_X^n$$

thus  $\mathbb{R}Sub_X$  is locally geometric.

### 3.4 Homotopy Flag Varieties and Derived Grassmannians

In this last section we will see how the ideas and notions discussed in Sections 3.2 and 3.3 allow us to construct homotopy versions of Grassmannians and flag varieties. Throughout

all this section fix our base scheme  $X$  to be the point  $\mathrm{Spec}(k)$  and let  $V$  be a bounded complex of finite-dimensional  $k$ -vector spaces; furthermore for all  $n \in \mathbb{N}$  consider the derived stacks

$$\begin{aligned}\mathbb{R}\mathrm{Perf}_k &:= \mathbb{R}\mathrm{Perf}_{\mathrm{Spec}(k)} & \mathbb{R}\mathrm{Perf}_k^n &:= \mathbb{R}\mathrm{Perf}_{\mathrm{Spec}(k)}^n \\ \mathbb{R}\mathrm{Filt}_k &:= \mathbb{R}\mathrm{Filt}_{\mathrm{Spec}(k)} & \mathbb{R}\mathrm{Filt}_k^n &:= \mathbb{R}\mathrm{Filt}_{\mathrm{Spec}(k)}^n \\ \mathbb{R}\mathrm{Sub}_k &:= \mathbb{R}\mathrm{Sub}_{\mathrm{Spec}(k)} & \mathbb{R}\mathrm{Sub}_k^n &:= \mathbb{R}\mathrm{Sub}_{\mathrm{Spec}(k)}^n\end{aligned}$$

which respectively parametrise:

- cochain complexes of  $k$ -vector spaces;
- filtered cochain complexes of  $k$ -vector spaces;
- pairs made of a cochain complex of  $k$ -vector spaces and a subcomplex.

**Definition 3.35** Define the *big total derived Grassmannian* over  $k$  associated to  $V$  to be the derived stack given by the homotopy fibre

$$\mathcal{DGRASS}_k(V) := \mathrm{holim}_{\leftarrow} \left( \mathbb{R}\mathrm{Sub}_k \xrightarrow[\mathrm{const}_V]{[F \hookrightarrow W] \mapsto W} \mathbb{R}\mathrm{Perf}_k \right)$$

where the top map is the natural forgetful morphism while “ $\mathrm{const}_V$ ” stands for the constant morphism sending any pair  $[F \hookrightarrow W]$  to  $V$ .

*Remark 3.36* The derived stack  $\mathcal{DGRASS}_k(W)$  is locally geometric: as a matter of fact we have that

$$\mathcal{DGRASS}_k(V) = \bigcup_n \mathcal{DGrass}_k^n(V)$$

where

$$\mathcal{DGrass}_k^n(V) := \mathrm{holim}_{\leftarrow} \left( \mathbb{R}\mathrm{Sub}_k^n \xrightarrow[\mathrm{const}_V]{[F \hookrightarrow W] \mapsto W} \mathbb{R}\mathrm{Perf}_k^n \right) \quad (3.44)$$

and formula (3.44) shows in particular that  $\mathcal{DGRASS}_k^n(V)$  is a derived geometric  $n$ -stack over  $k$ .

There is a more concrete realisation of the big total derived Grassmannian associated to  $V$ : indeed consider the functorial simplicial category

$$\begin{aligned}\forall A \in \mathcal{DgAlg}_k^{\leq 0} \quad DGRASS_k(V)(A) &:= \text{full simplicial subcategory made of sequences} \\ &U \hookrightarrow W \xrightarrow{\varphi} V \otimes A \\ &\text{of cofibrant } A\text{-modules in complexes} \\ &\text{where } \varphi \text{ is a quasi-isomorphism} \quad (3.45)\end{aligned}$$

and observe that  $\mathcal{DGRASS}_k(V) = \bar{W}W(DGRASS_k(V))$ .

Similarly, we can construct a preliminary derived notion of flag variety.

**Definition 3.37** Define the *big homotopy flag variety* over  $k$  associated to  $V$  to be the derived stack given by the homotopy fibre

$$\mathcal{DFLAG}_k(V) := \mathrm{holim}_{\leftarrow} \left( \mathbb{R}\mathrm{Filt}_k \xrightarrow[\mathrm{const}_V]{(W, F) \mapsto W} \mathbb{R}\mathrm{Perf}_k \right)$$

where the top map is the natural forgetful morphism while “ $\text{const}_V$ ” denotes again the constant morphism sending any filtered complex to  $V$ .

**Remark 3.38** The derived stack  $\mathcal{DFLAG}_k(V)$  is locally geometric: as a matter of fact we have that

$$\mathcal{DFLAG}_k(V) = \bigcup_n \mathcal{DFLAG}_k^n(V)$$

where

$$\mathcal{DFLAG}_k^n(V) := \text{holim}_{\leftarrow} \left( \mathbb{R}\mathcal{Filt}_k^n \xrightarrow[\text{const}_V]{(W,F) \mapsto W} \mathbb{R}\mathcal{P}erf_k^n \right) \quad (3.46)$$

and formula (3.46) shows in particular that  $\mathcal{DFLAG}_k^n(V)$  is a derived geometric  $n$ -stack over  $k$ .

There is a concrete realisation of the big homotopy flag variety given by equations similar to the ones supplied in formula (3.45) in the case of the big derived Grassmannian; as a matter of fact define the functorial simplicial category

$$\forall A \in \mathfrak{dgAlg}_k^{\leq 0} \quad \mathcal{DFLAG}_k(V)(A) := \text{full simplicial subcategory made of pairs } ((W, F), \varphi), \text{ with } (W, F) \text{ a filtered cofibrant } A\text{-module in complexes and } \varphi : W \rightarrow V \otimes A \text{ a quasi-isomorphism}$$

and observe that  $\mathcal{DFLAG}_k(V) = \bar{W}W(\mathcal{DFLAG}_k(V))$ .

**Remark 3.39** Assume that  $V$  is concentrated in degree 0 and consider the classical total Grassmannian variety

$$\begin{aligned} \text{Grass}(V) &:= \coprod_{i=0}^{\dim V} \text{Grass}(i, V) \\ \text{Grass}(i, V) &:= \{W \subseteq V \text{ s.t. } \dim(W) = i\}. \end{aligned} \quad (3.47)$$

We would like that the stack  $\mathcal{DGRASS}_k(V)$  were a derived enhancement of the variety (3.47), but unfortunately this is not the case as  $\mathcal{DGRASS}_k(V)$  is far too large: for instance, we have that  $\mathcal{DGRASS}_k(0) \approx \mathbb{R}\mathcal{P}erf_k$ ; analogous statements will hold for the flag variety  $\text{Flag}(V)$ .

Remark 3.39 tells us that the big total derived Grassmannian and the big homotopy flag variety are not derived enhancements of  $\text{Grass}(V)$  and  $\text{Flag}(V)$ ; anyhow hereinafter we will show that the two latter varieties can be realised respectively as the underived truncations of natural open substacks of  $\mathcal{DGRASS}_k(V)$  and  $\mathcal{DFLAG}_k(V)$ .

Consider the (underived) functorial simplicial category

$$\begin{aligned} \forall A \in \mathfrak{Alg}_k \quad \mathcal{DGrass}_k(V)(A) &:= \text{full simplicial subcategory of } \mathcal{DGRASS}_k(V)(A) \\ &\text{made of sequences } U \hookrightarrow W \xrightarrow{\phi} V \otimes A \\ &\text{for which } H^i(U) \text{ is flat over } A \\ &\text{and the induced morphism} \\ &H^i(U) \rightarrow H^i(V) \otimes A \\ &\text{is injective for all } i. \end{aligned}$$

as well as its enhancement

$$\begin{aligned} \forall A \in \mathfrak{dgAlg}_k^{\leq 0} \quad \widetilde{DGrass}_k(V)(A) &:= DGRASS_k(V)(A) \times_{DGRASS_k(V)(H^0(A))}^{(2)} DGrass_k(V)(H^0(A)) \\ &= \text{full simplicial subcategory of } DGRASS_k(V)(A) \\ &\quad \text{made of sequences } U \hookrightarrow W \xrightarrow{\phi} V \otimes A \\ &\quad \text{weakly equivalent to an object in } DGrass_k(V)(H^0(A)) \\ &\quad \text{after tensorisation with } H^0(A) \text{ over } A \end{aligned}$$

**Definition 3.40** For any cochain complex  $V$  define the *derived total Grassmannian associated to  $V$*  to be

$$\mathcal{DGrass}_k(V) := \bar{W}W(\widetilde{DGrass}_k(V)).$$

**Proposition 3.41**  $\mathcal{DGrass}_k(V)$  is an open derived substack of  $\mathcal{DGRASS}_k(V)$ .

*Proof* We want to show that the inclusion

$$i : \mathcal{DGrass}_k(V) \hookrightarrow \mathcal{DGRASS}_k(V)$$

is étale, which in turn amounts to prove that the induced map of formal groupoids

$$\pi_{\leq 0} DGrass_k(V) \longrightarrow \pi_{\leq 0} DGRASS_k(V)$$

is formally étale.

Let  $I \hookrightarrow A \twoheadrightarrow B$  be a square-zero extension in  $\mathfrak{Alg}_k$  and pick a triple  $[S \hookrightarrow W \twoheadrightarrow V \otimes B]$  in  $\pi_{\leq 0} DGrass_k(V)(B)$  – i.e. such that the induced morphism  $H^i(S) \rightarrow H^i(V) \otimes B$  stays injective for all  $i$  – and take  $[S' \hookrightarrow W' \twoheadrightarrow V \otimes A]$  in  $DGRASS_k(V)(A)$  such that  $S' \otimes_A B \approx S$  and  $W' \otimes_A B \approx W$ ; we need to show that the cohomology map  $H^i(S') \rightarrow H^i(V) \otimes A$  is injective. By taking long exact sequence in cohomology we end up with a morphism of complexes

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H^i(S) \otimes_B I & \longrightarrow & H^i(S') & \longrightarrow & H^i(S) \longrightarrow \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & H^i(V) \otimes I & \longrightarrow & H^i(V) \otimes A & \longrightarrow & H^i(V) \otimes B \longrightarrow \cdots \end{array} \quad (3.48)$$

in which the horizontal arrows are exact. Let  $v \in H^i(S')$  an element mapping to 0 in  $H^i(V) \otimes A$  and hence to 0 in  $H^i(V) \otimes B$ ; the injectivity of the map  $H^i(S) \rightarrow H^i(V) \otimes B$  implies that

$$v \in \ker(H^i(S') \rightarrow H^i(S)) \simeq \text{Im}(H^i(S) \otimes_B I \rightarrow H^i(S'))$$

so let  $w$  be a preimage of  $v$  inside  $H^i(S) \otimes_B I$ . Now let us walk along the commutative square on the left-hand side of diagram (3.48): we know that the (vertical) map

$$H^i(S) \otimes_B I \rightarrow H^i(V) \otimes B \otimes_B I \simeq H^i(V) \otimes I$$

is injective and notice furthermore that  $H^i(V)$  is flat over  $k$ ; as  $I \hookrightarrow A$ , it follows then that the (horizontal) map

$$H^i(V) \otimes I \simeq H^i(V) \otimes B \otimes_B I \rightarrow H^i(V) \otimes A$$



is injective, as well. As a result, we have that  $w$  is mapped to 0 in  $H^i(V) \otimes A$  via the composite of two injections, therefore  $w = 0$  and  $v = 0$ . This means that the map  $H^i(S') \rightarrow H^i(V) \otimes A$  is injective, which concludes the proof.  $\square$

**Theorem 3.42** *There is an isomorphism*

$$\pi^0 \pi_{\leq 0} \mathcal{D}\mathcal{G}rass_k(V) \simeq \mathbf{Grass}(H^*(V)) \quad (3.49)$$

where the right-hand side in formula (3.49) is the product of the classical total Grassmannians associated to the vector spaces  $H^i(V)$ ; in particular if  $V$  is concentrated in degree 0 then  $\mathcal{D}\mathcal{G}rass_k(V)$  is a derived enhancement of the classical total Grassmannian associated to  $V$ .

*Proof* We want to show that  $\pi^0 \pi_{\leq 0} \mathcal{D}\mathcal{G}rass_k(V)$  is the same as the functor of points represented by the variety  $\mathbf{Grass}(H^*(V))$ , which is

$$\begin{aligned} \mathbf{Grass}(H^*(V)) : \mathcal{A}lg_k &\longrightarrow \mathfrak{S}et \\ A &\longmapsto \{W \hookrightarrow H^*(V) \otimes A \text{ s.t. } W \text{ cofibrant}\}. \end{aligned}$$

Notice that, for all  $A \in \mathcal{A}lg_k$  we have that

$$\pi^0 \pi_{\leq 0} \mathcal{D}\mathcal{G}rass_k(V)(A) := \frac{\{T \hookrightarrow V \otimes A \text{ s.t. } T \text{ perfect, } H^*(T) \text{ } A\text{-flat, } H^*(T) \rightarrow H^*(V) \otimes A \text{ injective}\}}{\{\text{quasi-isomorphism}\}}.$$

Taking cohomology induces a natural bijection between the sets  $\pi^0 \pi_{\leq 0} \mathcal{D}\mathcal{G}rass_k(V)(A)$  and  $\mathbf{Grass}(H^*(V))(A)$ . Indeed consider  $[W \hookrightarrow H^*(V) \otimes A] \in \mathbf{Grass}(H^*(V))(A)$ : all we need to show is the existence and unicity of a quasi-isomorphism class

$$[T \hookrightarrow V \otimes A] \in \pi^0 \pi_{\leq 0} \mathcal{D}\mathcal{G}rass_k(V)(A)$$

whose cohomology is  $[W \hookrightarrow H^*(V) \otimes A]$ ; now this follows directly from the observation that the complex  $T$  is made of locally free modules in each degree, since it is perfect with flat cohomology.  $\square$

The constructions and results described by Definition 3.40, Proposition 3.41 and Theorem 3.42 for Grassmannians readily extend to the more general case of flag varieties.

Consider the (underived) functorial simplicial category

$$\begin{aligned} \forall A \in \mathcal{A}lg_k \quad DFlag_k(V)(A) &:= \text{full simplicial subcategory of } DFLAG_k(V)(A) \\ &\quad \text{made of pairs } ((W, F), \varphi) \\ &\quad \text{for which } H^j(F^i W) \text{ is flat over } A \\ &\quad \text{and the induced morphisms} \\ H^j(F^i W) &\rightarrow H^j(F^{i-1} W) \rightarrow H^*(V) \otimes A \\ &\quad \text{are injective for all } i, j. \end{aligned}$$

as well as its enhancement

$$\begin{aligned} \forall A \in \mathcal{D}g\mathcal{A}lg_k^{\leq 0} \quad \widetilde{DFlag}_V(A) &:= DFLAG_k(V)(A) \times_{DFLAG_k(V)(H^0(A))}^{(2)} DFlag_k(V)(H^0(A)) \\ &= \text{full simplicial subcategory of } DFLAG_k(V)(A) \\ &\quad \text{made of pairs } ((W, F), \varphi) \\ &\quad \text{weakly equivalent to an object in } DFlag_k(V)(H^0(A)) \\ &\quad \text{after tensorisation with } H^0(A) \text{ over } A \end{aligned}$$

**Definition 3.43** Define the *homotopy flag variety associated to  $V$*  to be

$$\mathcal{D}\mathcal{F}lag_k(V) := \bar{W}W(D\mathcal{F}lag_k(V)).$$

**Proposition 3.44**  $\mathcal{D}\mathcal{F}lag_k(V)$  is an open derived substack of  $\mathcal{D}\mathcal{F}\mathcal{L}\mathcal{A}\mathcal{G}_k(V)$ .

*Proof* The proof of Proposition 3.41 carries over to this context.  $\square$

**Theorem 3.45** The homotopy flag variety associated to  $V$  is a derived enhancement of the classical total flag variety attached to  $H^*(V)$ , i.e.

$$\pi^0_{\pi \leq 0} \mathcal{D}\mathcal{F}lag_k(V) \simeq \mathcal{F}lag(H^*(V)).$$

*Proof* The proof of Theorem 3.42 carries over to this context.  $\square$

**Remark 3.46** In this paper we have focused on the study of the global theory of Grassmannians and flag varieties in Derived Algebraic Geometry, ending up with the construction of  $\mathcal{D}\mathcal{G}rass_k(V)$  and  $\mathcal{D}\mathcal{F}lag_k(V)$ . The infinitesimal picture of these stacks – including the computation of their tangent complexes – will be analysed in [4].

## 4 Notations and Conventions

- $\text{diag}(-)$  = diagonal of a bisimplicial set
- $k$  = fixed field of characteristic 0, unless otherwise stated
- If  $A$  is a (possibly differential graded) local ring,  $\mathfrak{m}_A$  will be its unique maximal (possibly differential graded) ideal
- $R$  = (possibly differential graded) associative unital ring or  $k$ -algebra, unless otherwise stated; wherever  $R$  is also assumed to be commutative, it is specified in the body of the paper
- If  $(M, d)$  is a cochain complex (in some suitable category) then  $(M[n], d_{[n]})$  will be the cochain complex such that  $M[n]^j := M^{j+n}$  and  $d_{[n]}^j = d^{j+n}$
- $\mathbb{G}_m$  = multiplicative group scheme over  $k$
- $X$  = semi-separated quasi-compact scheme over  $R$  or  $k$  of finite dimension, unless otherwise stated; wherever  $X$  is also assumed to be smooth or proper, it is specified in the body of the paper
- $\mathcal{X}$  = derived scheme over  $R$
- $\mathcal{O}_X$  = structure sheaf of  $X$
- $\mathbb{L}^{\mathcal{F}/R}$  = (absolute) cotangent complex of the derived geometric stack  $\mathcal{F}$  over  $R$
- $D(X)$  = derived category of  $X$
- $\Delta$  = category of finite ordinal numbers
- $\mathcal{A}lg_k$  = category of commutative associative unital algebras over  $k$
- $\mathcal{A}lg_R$  = category of commutative associative unital algebras over  $R$
- $\mathcal{A}lg_{H^0(R)}$  = category of commutative associative unital algebras over  $H^0(R)$
- $\mathcal{C}h_{\geq 0}(\mathcal{V}ect_k)$  = model category of chain complexes of  $k$ -vector spaces
- $\mathcal{D}g\mathcal{A}lg_k^{\leq 0}$  = model category of (cochain) differential graded commutative algebras over  $k$  in non-positive degrees
- $\mathcal{D}g\mathcal{A}lg_R^{\leq 0}$  = model category of (cochain) differential graded commutative algebras over  $R$  in non-positive degrees

- $\mathrm{dg}\mathcal{A}lg_{R[t]}^{\leq 0}$  = model category of (cochain) differential graded commutative algebras over  $R[t]$  in non-positive degrees
- $\mathrm{dg}\mathcal{A}rt_k^{\leq 0}$  = model category of (cochain) differential graded local Artin algebras over  $k$  in non-positive degrees
- $\mathrm{dg}\mathcal{M}od_R$  = model category of  $R$ -modules in (cochain) complexes
- $\mathrm{dg}\mathcal{M}od(\mathcal{X})$  = model category of derived modules over  $\mathcal{X}$
- $\mathrm{dg}\mathcal{M}od(\mathcal{X})_{\mathrm{cart}}$  =  $\infty$ -category of derived quasi-coherent sheaves over  $\mathcal{X}$
- $\mathrm{dg}_b\mathcal{M}il_R^{\leq 0}$  =  $\infty$ -category of bounded below differential graded commutative  $R$ -algebras in non-positive degrees such that the canonical map  $A \rightarrow H^0(A)$  is nilpotent
- $\mathrm{dg}_b\mathcal{M}il_{H^0(R)}^{\leq 0}$  =  $\infty$ -category of bounded below differential graded commutative  $H^0(R)$ -algebras in non-positive degrees such that the canonical map  $A \rightarrow H^0(A)$  is nilpotent
- $\mathrm{dg}\mathcal{V}ect_k^{\leq 0}$  = model category of (cochain) differential graded vector spaces over  $k$  in non-positive degrees
- $\mathfrak{F}\mathrm{dg}\mathcal{M}od_R$  = model category of filtered  $R$ -modules in (cochain) complexes
- $\mathfrak{F}\mathrm{dg}\mathcal{M}od(\mathcal{X})$  = model category of filtered derived modules over  $\mathcal{X}$
- $\mathfrak{F}\mathrm{dg}\mathcal{M}od(\mathcal{X})_{\mathrm{cart}}$  =  $\infty$ -category of filtered derived quasi-coherent sheaves over  $\mathcal{X}$
- $\mathbb{G}_m\text{-}\mathrm{dg}\mathcal{M}od_{R[t]}$  = model category of graded  $R[t]$ -modules in (cochain) complexes
- $\mathbb{G}_m\text{-}\mathrm{dg}\mathcal{M}od(\mathcal{X}[t])$  = model category of graded derived modules over  $\mathcal{X}[t]$ <sup>16</sup>
- $\mathcal{G}\mathrm{rpd}$  = 2-category of groupoids
- $\mathcal{M}od_R$  = category of  $R$ -modules
- $\mathcal{P}\mathrm{erf}(X)$  = dg-category of perfect complexes of  $\mathcal{O}_X$ -modules
- $\mathcal{Q}\mathrm{Coh}(X)$  = category of quasi-coherent sheaves over  $X$
- $\mathcal{S}\mathrm{et}$  = category of sets
- $\mathfrak{s}\mathcal{A}lg_k$  = model category of simplicial commutative associative unital algebras over  $k$
- $\mathfrak{s}\mathcal{C}\mathrm{at}$  = model category of simplicial categories
- $\mathfrak{s}\mathcal{S}\mathrm{et}$  = simplicial model category of simplicial sets
- $\mathfrak{s}\mathcal{V}\mathrm{ect}_k$  = model category of simplicial vector spaces over  $k$
- $\mathcal{V}\mathrm{ect}_k$  = category of vector spaces over  $k$

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<sup>16</sup>The definition of the derived scheme  $\mathcal{X}[t]$  is given in the body of the paper (Section 3.3).

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